The Method of Undetermined Coefficients for Forcing Functions that Solve the Homogeneous Equation

Bernd Schröder

The Method of Undetermined Coefficients

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2. In the Method of Undetermined Coefficients we detect repeating patterns in the derivatives of the inhomogeneity and set up the particular solution as a linear combination of the patterns with undetermined coefficients.
3. The conjectured solution is substituted into the equation to determine the coefficients.
4. When the forcing function is a solution of the homogeneous equation, multiply it with the independent variable until it no longer solves the homogeneous equation. Start your pattern with that function.

Solve the Differential Equation $y'' + 9y = \sin(3x)$

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Need to multiply the right side by x to get the function that starts the pattern.

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Generating the *form* of the particular solution of the inhomogeneous equation.

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r(x) &= x\sin(3x) && \text{(need a term } x\sin(3x)\text{)} \\
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y_p &:= Ax\cos(3x) + Bx\sin(3x)
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y_p'' &= -9Ax\cos(3x) - 3A\sin(3x) - 3A\sin(3x) \\
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&= -9Ax\cos(3x) - 6A\sin(3x) - 9Bx\sin(3x) + 6B\cos(3x)
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y''+9y &= \sin(3x) \\
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(-9A+9A)x\cos(3x)+(-9B+9B)x\sin(3x) & \\
+(6B)\cos(3x)+(-6A)\sin(3x) &= \sin(3x) \\
6B=0,\quad B=0,\qquad -6A=1,\quad A=-\frac{1}{6} & \\
y_p &= -\frac{1}{6}x\cos(3x)
\end{aligned}$$

$$y=-\frac{1}{6}x\cos(3x)+c_1\cos(3x)+c_2\sin(3x)$$

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y_p'' = \frac{3}{2}x\cos(3x) + \frac{1}{2}\sin(3x) + \frac{1}{2}\sin(3x) &= \frac{3}{2}x\cos(3x) + \sin(3x) \\
y'' + 9y &\stackrel{?}{=} \sin(3x) \\
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\end{aligned}$$

What would have happened if we had used $\sin(3x)$ or $\cos(3x)$?

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$$y = A\sin(3x), \quad y' = 3A\cos(3x), \quad y'' = -9A\sin(3x)$$

$$\begin{aligned} y'' + 9y &= \sin(3x) \\ -9A\sin(3x) + 9A\sin(3x) &= \sin(3x) \\ 0 &= \sin(3x) \end{aligned}$$

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Whenever you set up the Method of Undetermined Coefficients and something zeros out, double check if you started with a solution of the homogeneous equation and adjust appropriately.

How to Generate the Setup When the Forcing Function Solves the Homogeneous Equation

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1. The starting term is the forcing function, adjusted by multiplying by powers of x, as demonstrated here.
2. Compute derivatives of the starting term until no new patterns emerge.
3. Generate a term (with an undetermined coefficient) for every new term that occurs in one of the derivatives, *except* for solutions of the homogeneous equation.

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$$y'' + 2y' + y = e^{-x}$$

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$$\begin{aligned} y''+2y'+y &= e^{-x} \\ y_h &= c_1e^{-x}+c_2xe^{-x} \end{aligned}$$

How to Generate the Setup When the Forcing Function Solves the Homogeneous Equation

$$
\begin{aligned}
y'' + 2y' + y &= e^{-x} \\
y_h &= c_1 e^{-x} + c_2 x e^{-x} \\
r(x) &= e^{-x}
\end{aligned}
$$

How to Generate the Setup When the Forcing Function Solves the Homogeneous Equation

$$
\begin{aligned}
y''+2y'+y &= e^{-x}\\
y_h &= c_1e^{-x}+c_2xe^{-x}\\
r(x) &= e^{-x} \qquad \text{(solves the homogeneous equation)}
\end{aligned}
$$

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$$\begin{aligned} y'' + 2y' + y &= e^{-x} \\ y_h &= c_1 e^{-x} + c_2 x e^{-x} \\ r(x) &= e^{-x} \end{aligned}$$

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y_p &= Ax^2 e^{-x}
\end{aligned}
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Resonance

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$$y'' + 9y = \sin(3x)$$

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