Integral Theorems

Bernd Schröder
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6. The original motivation to investigate integrals over closed contours probably comes from considerations of potentials in physics. For potentials in physics, integrals over closed curves must be zero.
**Theorem.**

Let $f$ be a continuous complex function on a domain $D$. The following are equivalent.

1. The function $f$ has an antiderivative $F$ on $D$. (That is, $F'(z) = f(z)$ for all $z$ in $D$.)
2. The integrals of $f$ over any contour $C$ from $z_1$ to $z_2$ in $D$ only depend on $z_1$ and $z_2$, but not on $C$ itself.
3. For any closed contour in $D$ we have that $\int_C f(z) \, dz = 0$.

In the above situation, if $C$ is a contour from $z_1$ to $z_2$, then $\int_C f(z) \, dz = F(z_2) - F(z_1)$. 

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(Also see earlier presentation for the direct computation for the unit circle.)
Example.

The integral of the function $z^{-1}$ around the unit circle is $2\pi i$. It's tempting to declare the logarithm an antiderivative of $z^{-1}$. But because of the problems with defining a logarithm function on a deleted neighborhood of zero (it's not possible, that's why we work with branches), the logarithm is not an antiderivative of $z^{-1}$ on any deleted neighborhood of zero. Also recall $\int_C z^{-1} \, dz = \int_{0}^{2\pi} (e^{it})^{-1} i e^{it} \, dt = \int_{0}^{2\pi} i \, dt = 2\pi i$. The logarithm is an antiderivative of $z^{-1}$ on any subset of the complex numbers from which an appropriate branch cut has been removed, though.
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**Proof (”1⇒3”).** If $F$ is an antiderivative of $f$ and $C$ is a closed contour from $z(a)$ to $z(b) = z(a)$, then

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Note that $C := C_1 + (-C_2)$ is a closed contour. 

$$0 = \int_{C} f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz$$ 

Thus the integrals along any contour from $z_1$ to $z_2$ all have the same value, which means that, for arbitrary $z_1$ and $z_2$, the integral only depends on $z_1$ and $z_2$, not on the path we take from one point to the other.
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Proof ("2⇒1"). Fix a point $z_0$ in $D$. For $z$ in $D$ (recall that $D$ is connected) define $F(z) := \int_{z_0}^{z} f(\gamma) \, d\gamma$. Because the integral only depends on the endpoints, we need not specify the contour, and in this case it is common to use notation that is similar to that for integrals over intervals on the real line. We claim that $F' = f$. Let $z$ be in $D$.

$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z} = \lim_{w \to z} \frac{1}{w - z} \left( \int_{z_0}^{w} f(\gamma) \, d\gamma - \int_{z_0}^{z} f(\gamma) \, d\gamma \right)$$

$$= \lim_{w \to z} \frac{1}{w - z} \int_{z}^{w} f(\gamma) \, d\gamma = f(z)$$

because, with the contour from $z$ to $w$ chosen to be a straight line, as $w \to z$, the values $f(\gamma)$ are close to $f(z)$, so that the integral is close ("and in the limit equal") to $f(z)(w - z)$.
Proof (finish).
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Theorem.
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\[
\int_{[a, b, c, a]} f(z) \, dz = \int_{[a, m_{ab}, m_{ca}, a]} f(z) \, dz + \int_{[b, m_{bc}, m_{ab}, b]} f(z) \, dz + \int_{[c, m_{ca}, m_{bc}, c]} f(z) \, dz + \int_{[m_{ab}, m_{bc}, m_{ca}, m_{ab}]} f(z) \, dz.
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\[
\int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},b,m_{bc},c,m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},a]} f(z) \, dz + \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz.
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\[
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\[
\int_{[a, b, c, a]} f(z) \, dz = \int_{[a, m_{ab}, m_{ca}, a]} f(z) \, dz + \int_{[b, m_{bc}, m_{ab}, b]} f(z) \, dz
\]
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\[
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\[ \int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz \]
\[ + \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz. \]
Proof.
Proof.

\[
\int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz
\]
\[
+ \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz
\]
Proof.

\[
\int_{[a,b,c,a]} f(z) \, dz \quad = \quad \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz \\
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\]

implies that the absolute value of one of the integrals on the right is greater than or equal to \( \frac{1}{4} \int_{[a,b,c,a]} f(z) \, dz \). Thus, if \( a_0 := a \)
Proof.

\[
\int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz \\
+ \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz
\]

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Proof.

\[
\int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz \\
+ \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz
\]

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Proof.
\[
\int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz \\
+ \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz
\]
implies that the absolute value of one of the integrals on the right is greater than or equal to \( \frac{1}{4} \left| \int_{[a,b,c,a]} f(z) \, dz \right| \). Thus, if \( a_0 := a \), \( b_0 := b \), \( c_0 := c \), then we can find \( a_1, b_1, c_1 \) so that
\[
\left| \int_{[a_1,b_1,c_1,a_1]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_0,b_0,c_0,a_0]} f(z) \, dz \right|
\]
Proof.

\[
\int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz \\
+ \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz
\]

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\[
\left| \int_{[a_1,b_1,c_1,a_1]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_0,b_0,c_0,a_0]} f(z) \, dz \right|
\]

so that the lengths of the sides satisfy

\[
l([a_1, b_1, c_1, a_1]) \leq \frac{1}{2} \cdot l([a_0, b_0, c_0, a_0])
\]
Proof.

\[
\int_{[a,b,c,a]} f(z) \, dz = \int_{[a,m_{ab},m_{ca},a]} f(z) \, dz + \int_{[b,m_{bc},m_{ab},b]} f(z) \, dz
\]

\[
+ \int_{[c,m_{ca},m_{bc},b]} f(z) \, dz + \int_{[m_{ab},m_{bc},m_{ca},m_{ab}]} f(z) \, dz
\]

implies that the absolute value of one of the integrals on the right is greater than or equal to \( \left| \frac{1}{4} \int_{[a,b,c,a]} f(z) \, dz \right| \). Thus, if \( a_0 := a, b_0 := b, c_0 := c \), then we can find \( a_1, b_1, c_1 \) so that

\[
\left| \int_{[a_1,b_1,c_1,a_1]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_0,b_0,c_0,a_0]} f(z) \, dz \right|
\]

so that the lengths of the sides satisfy

\[
l([a_1, b_1, c_1, a_1]) \leq \frac{1}{2} l([a_0, b_0, c_0, a_0])\]

and so that the diameters satisfy

\[
diam(\Delta(a_1, b_1, c_1)) \leq \frac{1}{2} diam(\Delta(a_0, b_0, c_0)).\]
Proof.
Proof.
Proof.

\[
\begin{align*}
&|\int_{[a_n, b_n, c_n, a_n]} f(z) \, dz| \\
\geq &\cdot \cdot \cdot \\
\geq & 1/4 |\int_{[a_0, b_0, c_0, a_0]} f(z) \, dz| \\
\leq &\cdot \cdot \cdot \\
\leq & 1/2 \text{diam}(\Delta(a_n, b_n, c_n)) \\
\leq &\cdot \cdot \cdot \\
\leq & 1/2 n \text{diam}(\Delta(a_0, b_0, c_0))
\end{align*}
\]
Proof.
Proof.

\[
\int_{\gamma_0} f(z) \, dz \geq \frac{1}{4} \left( \int_{\gamma_{n-1}} f(z) \, dz \right) \geq \cdots \geq \frac{1}{2^n} \left( \int_{\gamma_0} f(z) \, dz \right)
\]

\[
\text{diam} \left( \Delta(a_n, b_n, c_n) \right) \leq \frac{1}{2} \text{diam} \left( \Delta(a_{n-1}, b_{n-1}, c_{n-1}) \right) \leq \cdots \leq \frac{1}{2^n} \text{diam} \left( \Delta(a_0, b_0, c_0) \right)
\]
Proof.
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Proof.
**Proof.**
Proof.

\[
\left| \int_{\gamma_n} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{\gamma_{n-1}} f(z) \, dz \right| \geq \cdots \geq \frac{1}{2^n} \left| \int_{\gamma_0} f(z) \, dz \right|
\]

\[
\text{diam}(\Delta(a_n, b_n, c_n)) \leq \frac{1}{2} \text{diam}(\Delta(a_{n-1}, b_{n-1}, c_{n-1})) \leq \cdots \leq \frac{1}{2^n} \text{diam}(\Delta(a_0, b_0, c_0))
\]
Proof.

\[
\left| \int_{[a_n, b_n, c_n, a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]} f(z) \, dz \right| \geq \cdots \geq \frac{1}{2^n} \left| \int_{[a_0, b_0, c_0, a_0]} f(z) \, dz \right| \leq \frac{1}{2^n} \text{diam} \left( \Delta (a_n, b_n, c_n) \right) \leq \cdots \leq \frac{1}{2} \text{diam} \left( \Delta (a_0, b_0, c_0) \right).
\]
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\[
\left| \int_{[a_n, b_n, c_n, a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]} f(z) \, dz \right|
\]
Proof.

\[
\left| \int_{[a_n, b_n, c_n, a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]} f(z) \, dz \right| \geq \cdots
\]
Proof.

\[
\left| \int_{[a_n,b_n,c_n,a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1},b_{n-1},c_{n-1},a_{n-1}]} f(z) \, dz \right| \geq \cdots \\
\geq \frac{1}{4^n} \left| \int_{[a_0,b_0,c_0,a_0]} f(z) \, dz \right|
\]
Proof.

\[
\left| \int_{[a_n, b_n, c_n, a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]} f(z) \, dz \right| \geq \cdots
\]

\[
\geq \frac{1}{4^n} \left| \int_{[a_0, b_0, c_0, a_0]} f(z) \, dz \right|
\]

\[
l([a_n, b_n, c_n, a_n]) \leq \frac{1}{2} l([a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}])
\]
Proof.

\[
\left| \int_{[a_n, b_n, c_n, a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]} f(z) \, dz \right| \geq \cdots \\
\geq \frac{1}{4^n} \left| \int_{[a_0, b_0, c_0, a_0]} f(z) \, dz \right| \\
l([a_n, b_n, c_n, a_n]) \leq \frac{1}{2} l([a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]) \leq \cdots
\]
Proof.

\[
\left| \int_{[a_n,b_n,c_n,a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1},b_{n-1},c_{n-1},a_{n-1}]} f(z) \, dz \right| \geq \cdots \\
\geq \frac{1}{4^n} \left| \int_{[a_0,b_0,c_0,a_0]} f(z) \, dz \right| \\
l([a_n,b_n,c_n,a_n]) \leq \frac{1}{2} l([a_{n-1},b_{n-1},c_{n-1},a_{n-1}]) \leq \cdots \leq \frac{1}{2^n} l([a_0,b_0,c_0,a_0])
\]
Proof.

\[ \left| \int_{[a_n, b_n, c_n, a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]} f(z) \, dz \right| \geq \cdots \]

\[ \geq \frac{1}{4^n} \left| \int_{[a_0, b_0, c_0, a_0]} f(z) \, dz \right| \]

\[ l([a_n, b_n, c_n, a_n]) \leq \frac{1}{2} l([a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]) \leq \cdots \leq \frac{1}{2^n} l([a_0, b_0, c_0, a_0]) \]

\[ \text{diam} \left( \Delta(a_n, b_n, c_n) \right) \leq \frac{1}{2} \text{diam} \left( \Delta(a_{n-1}, b_{n-1}, c_{n-1}) \right) \]
Proof.

\[ \left| \int_{[a_n,b_n,c_n,a_n]} f(z) \, dz \right| \geq \frac{1}{4} \left| \int_{[a_{n-1},b_{n-1},c_{n-1},a_{n-1}]} f(z) \, dz \right| \geq \cdots \]

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\[ \text{diam}(\Delta(a_n,b_n,c_n)) \leq \frac{1}{2} \text{diam}(\Delta(a_{n-1},b_{n-1},c_{n-1})) \leq \cdots \]
Proof.

\[
\begin{align*}
\left| \int_{[a_n, b_n, c_n, a_n]} f(z) \, dz \right| & \geq \frac{1}{4} \left| \int_{[a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]} f(z) \, dz \right| \geq \cdots \\
& \geq \frac{1}{4^n} \left| \int_{[a_0, b_0, c_0, a_0]} f(z) \, dz \right|
\end{align*}
\]

\[
\begin{align*}
l([a_n, b_n, c_n, a_n]) & \leq \frac{1}{2} l([a_{n-1}, b_{n-1}, c_{n-1}, a_{n-1}]) \leq \cdots \leq \frac{1}{2^n} l([a_0, b_0, c_0, a_0]) \\
diam(\Delta(a_n, b_n, c_n)) & \leq \frac{1}{2} diam(\Delta(a_{n-1}, b_{n-1}, c_{n-1})) \leq \cdots \leq \frac{1}{2^n} diam(\Delta(a_0, b_0, c_0))
\end{align*}
\]
Proof.
Proof. By definition of the derivative

\[ 0 = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \]
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\[ 0 = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) =: \lim_{z \to z_0} h(z) \]
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Thus there is a function \( h \) so that for all \( z \) we have

\[ f(z) = f(z_0) + f'(z_0)(z - z_0) + h(z)(z - z_0) \]

and \( \lim_{z \to z_0} h(z) = 0 \). For any \( \varepsilon > 0 \) we can find an \( n \) so that

\[ \sup_{z \in [a_n, b_n, c_n, a_n]} |h(z)| < \frac{\varepsilon}{l[a, b, c, a] \text{diam}(\Delta(a, b, c))}. \]
Proof.
**Proof.** Then

\[\left| \int_{[a,b,c,a]} f(z) \, dz \right|\]
**Proof.** Then

\[
\left| \int_{[a,b,c,a]} f(z) \, dz \right| \leq 4^n \left| \int_{[a_n,b_n,c_n,a_n]} f(z) \, dz \right|
\]
Proof. Then
\[
\left| \int_{[a,b,c,a]} f(z) \, dz \right| \leq 4^n \left| \int_{[a_n,b_n,c_n,a_n]} f(z) \, dz \right|
\]
\[
= 4^n \left| \int_{[a_n,b_n,c_n,a_n]} f(z_0) + f'(z_0)(z - z_0) + h(z)(z - z_0) \, dz \right|
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\[
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\]

\[
= 4^n \left| \int_{[a_n,b_n,c_n,a_n]} h(z)(z - z_0) \, dz \right| \leq 4^n \int_{[a_n,b_n,c_n,a_n]} |z - z_0| |h(z)| \, d|z|
\]
Proof. Then

\[
\left| \int_{[a,b,c,a]} f(z) \, dz \right| \leq 4^n \left| \int_{[a_n,b_n,c_n,a_n]} f(z) \, dz \right|
\]

\[
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\[
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\]

\[
\leq 4^n l[a_n, b_n, c_n, a_n] \text{diam} \left( \Delta(a_n, b_n, c_n) \right) \sup_{z \in [a_n, b_n, c_n, a_n]} |h(z)|
\]
Proof. Then

\[ \left| \int_{[a,b,c,a]} f(z) \, dz \right| \leq 4^n \left| \int_{[a_n,b_n,c_n,a_n]} f(z) \, dz \right| \]

\[ = 4^n \left| \int_{[a_n,b_n,c_n,a_n]} f(z_0) + f'(z_0)(z - z_0) + h(z)(z - z_0) \, dz \right| \]

\[ = 4^n \left| \int_{[a_n,b_n,c_n,a_n]} h(z)(z - z_0) \, dz \right| \leq 4^n \int_{[a_n,b_n,c_n,a_n]} |(z - z_0)| |h(z)| \, d|z| \]

\[ \leq 4^n L[a_n,b_n,c_n,a_n] \text{diam}\left(\Delta(a_n,b_n,c_n)\right) \sup_{z \in [a_n,b_n,c_n,a_n]} |h(z)| \]

\[ = 4^n \frac{1}{2^n} L[a,b,c,a] \frac{1}{2^n} \text{diam}\left(\Delta(a,b,c)\right) \sup_{z \in [a_n,b_n,c_n,a_n]} |h(z)| \]
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$$
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$$

$$
= 4^n \left| \int_{[a_n,b_n,c_n,a_n]} f(z_0) + \frac{d}{dz} f(z_0) (z - z_0) + h(z)(z - z_0) \, dz \right|
$$

$$
= 4^n \left| \int_{[a_n,b_n,c_n,a_n]} h(z)(z - z_0) \, dz \right| \leq 4^n \int_{[a_n,b_n,c_n,a_n]} \left| z - z_0 \right| \left| h(z) \right| \, d|z|
$$

$$
\leq 4^n l[a_n,b_n,c_n,a_n] \text{diam}\left(\Delta(a_n,b_n,c_n)\right) \sup_{z \in [a_n,b_n,c_n,a_n]} \left| h(z) \right|
$$

$$
= 4^n \frac{1}{2^n} l[a,b,c,a] \frac{1}{2^n} \text{diam}\left(\Delta(a,b,c)\right) \sup_{z \in [a_n,b_n,c_n,a_n]} \left| h(z) \right|
$$

$$
\leq l[a,b,c,a] \text{diam}\left(\Delta(a,b,c)\right) \sup_{z \in [a_n,b_n,c_n,a_n]} \left| h(z) \right| < \varepsilon
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\[ \leq 4^n l[a_n, b_n, c_n, a_n] \text{diam} \left( \Delta(a_n, b_n, c_n) \right) \sup_{z \in [a_n, b_n, c_n, a_n]} \left| h(z) \right| \]

\[ = 4^n \frac{1}{2^n} l[a, b, c, a] \frac{1}{2^n} \text{diam} \left( \Delta(a, b, c) \right) \sup_{z \in [a_n, b_n, c_n, a_n]} \left| h(z) \right| \]

\[ \leq l[a, b, c, a] \text{diam} \left( \Delta(a, b, c) \right) \sup_{z \in [a_n, b_n, c_n, a_n]} \left| h(z) \right| < \varepsilon \]

implies that \( \int_{[a, b, c, a]} f(z) \, dz = 0 \), because \( \varepsilon \) was arbitrary.

Bernd Schröder
Louisiana Tech University, College of Engineering and Science
Integral Theorems
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Proof.
Domains With and Without Holes

1. The Cauchy-Goursat Theorem works as long as the function is analytic on a domain that contains the contour and the contour's interior.

2. But if \( C \) is the positively oriented unit circle, then \( \int_C z^{-1} \, dz = 2\pi i \neq 0 \) shows that the result need not hold when the function is not analytic in the whole interior.

3. Side note: \( \int_C z^{-2} \, dz = 0 \) shows that just because the function is not analytic in the interior, the theorem need not fail automatically.

This is quite common in mathematics and in life. If your hypotheses are satisfied, then you can say something with confidence. But if not, it's often "anything goes."
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Domains With and Without Holes

4. The problem with $z - 1$ on the unit circle is apparently that the function is not analytic (not even defined) at $z = 0$.

5. Pictorially, the domain of $z - 1$ has a "hole" at zero, and we want to formalize that idea.
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5. Pictorially, the domain of $z^{-1}$ has a “hole” at zero, and we want to formalize that idea.
Definition.
Definition. A domain $D$ is called simply connected.

Simply connected means "no holes".
Definition. A domain $D$ *(remember domains are connected)*
**Definition.** A domain $D$ (remember domains are connected) so that for every simple closed contour $C$ in $D$ the interior of $C$ is contained in $D$, too, is called **simply connected**.
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Theorem.
**Theorem.** If $f$ is analytic on the simply connected domain $D$, then for any closed contour $C$ in $D$ we have $\int_C f(z) \, dz = 0$. 
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**Proof.**
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Proof. By the preceding theorem, integrals of $f$ over closed contours are zero.
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**Theorem.** Let $C$ be a positively oriented simple closed contour. Let $C_1, \ldots, C_n$ be pairwise disjoint clockwise (that is, negatively) oriented simple closed contours in the interior of $C$. Let $f$ be analytic in a (possibly multiply connected) domain that contains the contours and the region inside $C$ and outside the $C_j$. 
**Theorem.** Let $C$ be a positively oriented simple closed contour. Let $C_1, \ldots, C_n$ be pairwise disjoint clockwise (that is, negatively) oriented simple closed contours in the interior of $C$. Let $f$ be analytic in a (possibly multiply connected) domain that contains the contours and the region inside $C$ and outside the $C_j$. Then

$$\int_C f(z) \, dz + \sum_{j=1}^{n} \int_{C_j} f(z) \, dz = 0.$$
Proof.
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Proof.
Theorem.

Let $C_1$ and $C_2$ be positively oriented simple closed contours so that $C_1$ is contained in the interior of $C_2$. Let $f$ be analytic in a region that contains the contours and the region between them. Then

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz.$$ 

Proof. Same proof as previous result, except that, because both contours are positively oriented, this time the difference is zero.

Bring the integral over $C_2$ to the right side.
**Theorem.** Let $C_1$ and $C_2$ be positively oriented simple closed contours so that $C_1$ is contained in the interior of $C_2$. 
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**Theorem.** Let $C_1$ and $C_2$ be positively oriented simple closed contours so that $C_1$ is contained in the interior of $C_2$. Let $f$ be analytic in a region that contains the contours and the region between them. Then $\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz$. 
Theorem. Let $C_1$ and $C_2$ be positively oriented simple closed contours so that $C_1$ is contained in the interior of $C_2$. Let $f$ be analytic in a region that contains the contours and the region between them. Then \[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz. \]

Proof.
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Example.

The integral of \( \frac{1}{z} \) around any positively oriented simple closed contour that has the origin in its interior is \( 2\pi i \).

We have proved that the integral of this function along positively oriented circles around the origin is \( 2\pi i \).

The preceding theorem lets us go to arbitrary positively oriented simple closed contours that have the origin in their interior, because we can always put a tiny circle into the contour's interior or draw a large circle around it.
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