Series Expansion of Analytic Functions

Bernd Schröder
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4. This result will help simplify a lot of later results.
5. It is one of the reasons why complex analysis is so powerful: Analytic functions are very well-behaved.
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5. The goal is to get to the power series expansion of analytic functions as efficiently as possible.
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![Diagram of complex plane with point $z_1$]
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![Diagram showing complex numbers and their limits](image-url)
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Theorem. Let $L = L_x + iL_y$ and let $z_n = x_n + iy_n$. 

Proof. Let $\varepsilon > 0$. If $\lim_{n \to \infty} z_n = L$ then there is an integer $N$ so that for all $n \geq N$ we have $|z_n - L| < \varepsilon$. But then for all $n \geq N$ we have $|x_n - Lx| < \varepsilon$. Conversely, if $\lim_{n \to \infty} x_n = Lx$ and $\lim_{n \to \infty} y_n = Ly$ then there is an integer $N$ so that for all $n \geq N$ we have $|x_n - Lx| < \epsilon^2$ and $|y_n - Ly| < \epsilon^2$. But then for all $n \geq N$ we have $|z_n - L| = |(x_n + iy_n) - (Lx + iLy)| \leq |x_n - Lx| + |y_n - Ly| < \epsilon^2 + \epsilon^2$. 

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**Theorem.** Let \( L = L_x + iL_y \) and let \( z_n = x_n + iy_n \). Then \( \lim_{n \to \infty} z_n = L \) if and only if

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Theorem. Let $L = L_x + iL_y$ and let $z_n = x_n + iy_n$. Then $\lim_{n \to \infty} z_n = L$ if and only if $\lim_{n \to \infty} x_n = L_x$ and $\lim_{n \to \infty} y_n = L_y$. 
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**Proof.** Let $\varepsilon > 0$. If $\lim_{n \to \infty} z_n = L$ then there is an integer $N$ so that for all $n \geq N$ we have $|z_n - L| < \varepsilon$. 
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Example.
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\[ \lim_{n \to \infty} \frac{1}{n} + i \frac{n^2 + 1}{n^2} = (\lim_{n \to \infty} \frac{1}{n}) + i (\lim_{n \to \infty} \frac{1}{n} + \frac{1}{n^2}) = 0 + i \cdot 1 = i. \]
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Example. For $|z| < 1$ we have $\lim_{n \to \infty} \frac{z^n}{1 - z} = 0$

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\lim_{n \to \infty} \frac{z^n}{1 - z} = \lim_{n \to \infty} \frac{z^n (1 - \bar{z})}{(1 - z)(1 - \bar{z})} = \lim_{n \to \infty} \frac{r^n e^{in\theta} (1 - re^{-i\theta})}{1 - 2\Re(z) + |z|^2} = \lim_{n \to \infty} \frac{r^n (\cos(n\theta) + i\sin(n\theta)) (1 - r\cos(\theta) + ir\sin(\theta))}{1 - 2\Re(z) + |z|^2}
\]

\[
= \lim_{n \to \infty} \frac{r^n (\cos(n\theta)(1 - r\cos(\theta)) - r\sin(n\theta)\sin(\theta))}{1 - 2\Re(z) + |z|^2} + \lim_{n \to \infty} i\frac{r^n (\cos(n\theta)r\sin(\theta) + \sin(n\theta)(1 - r\cos(\theta)))}{1 - 2\Re(z) + |z|^2}
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\[= 0\]
Definition.
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Definition. An infinite series $z_1 + z_2 + z_3 + \cdots = \sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge to a limit $L$ if and only if

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Definition. An infinite series $z_1 + z_2 + z_3 + \cdots =: \sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge to a limit $L$ if and only if the sequence of partial sums $\sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N$ converges to $L$. 
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**Definition.** An infinite series $z_1 + z_2 + z_3 + \cdots = \sum_{n=1}^{\infty} z_n$ of complex numbers is said to **converge** to a limit $L$ if and only if the sequence of partial sums $\sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N$ converges to $L$. In this case we write $\sum_{n=1}^{\infty} z_n = L$.

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2. Also note that technically we really have not defined what a series is. And that’s quite common. (I’ve looked.)
Theorem.
**Theorem.** Let $L = L_x + iL_y$ and let $z_n = x_n + iy_n$. 

Proof. Let $Z_N = \sum_{n=1}^{N} z_n$, $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$. Now apply the earlier theorem on sequences to the sequences denoted with the capital letters.
**Theorem.** Let \( L = L_x + iL_y \) and let \( z_n = x_n + iy_n \). Then \( \sum_{n=1}^{\infty} z_n = L \) if and only if

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\sum_{n=1}^{\infty} x_n = L_x \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = L_y.
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Bernd Schröder
Louisiana Tech University, College of Engineering and Science
Series Expansion of Analytic Functions
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Theorem.

Let $L = \sum_{n=1}^{\infty} z_n$ and let $\varepsilon > 0$. Then there is an integer $N$ so that for all $k \geq N$ we have $|L - k - 1 \sum_{n=1}^{k} z_n| < \varepsilon/2$. Thus for all $k \geq N$ we have $|z_k| = |k \sum_{n=1}^{k} z_n - L - k \sum_{n=1}^{k-1} z_n| \leq |k \sum_{n=1}^{k} z_n - L| + |k \sum_{n=1}^{k-1} z_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus $\lim_{k \to \infty} z_k = 0$. 
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\left| L - \sum_{n=1}^{k-1} z_n \right| < \frac{\varepsilon}{2}.
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\]

\[
\leq \left| \sum_{n=1}^{k} z_n - L \right| + \left| \sum_{n=1}^{k-1} z_n - L \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
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Thus $\lim_{k\to\infty} z_k = 0$.  \[\blacksquare\]
WARNING: The converse is NOT true.

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \cdots \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \]
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\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1
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\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} \]
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\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots
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\[ + \frac{1}{3} + \frac{1}{4} \]

\[ + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \]

\[ + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \]
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1. Absolute convergence strips away everything that is hard about complex numbers and reduces our series to a series of nonnegative numbers.
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Series Expansion of Analytic Functions
**Definition.** A series \( \sum_{n=1}^{\infty} z_n \) of complex numbers is said to **converge absolutely** if and only if \( \sum_{n=1}^{\infty} |z_n| \) converges.

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Definition. A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge absolutely if and only if $\sum_{n=1}^{\infty} |z_n|$ converges.

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\]

\[
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1. Absolute convergence strips away everything that is hard about complex numbers and reduces our series to a series of nonnegative numbers.

2. Unfortunately not every convergent series is absolutely convergent: 

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\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{n}{2}} = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots = 0, \text{ but }
\sum_{n=1}^{\infty} \frac{1}{\binom{n}{2}} \text{ which is larger than } \sum_{n=1}^{\infty} \frac{1}{n}, \text{ which diverges.}
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**Proof.**

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Example. For all complex numbers $q$ with $|q| < 1$ we have that
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\begin{align*}
\sum_{n=0}^{\infty} q^n &= \lim_{N \to \infty} \sum_{n=0}^{N} q^n \\
&= \lim_{N \to \infty} \frac{1 - q^{N+1}}{1 - q} \\
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$$

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Theorem.
Theorem. Let $f$ be analytic on a disk $|z - z_0| \leq R$. 
**Theorem.** Let $f$ be analytic on a disk $|z - z_0| \leq R$. Then there is a sequence $a_n$ of complex numbers so that for every $z$ in the disk we have that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. 

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Series Expansion of Analytic Functions
**Theorem.** Let \( f \) be analytic on a disk \( |z - z_0| \leq R \). Then there is a sequence \( a_n \) of complex numbers so that for every \( z \) in the disk we have that
\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.
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Moreover, \( a_n = \frac{f^{(n)}(z_0)}{n!} \).
Theorem. Let $f$ be analytic on a disk $|z - z_0| \leq R$. Then there is a sequence $a_n$ of complex numbers so that for every $z$ in the disk we have that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. Moreover, $a_n = \frac{f^{(n)}(z_0)}{n!}$. 
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Series Expansion of Analytic Functions
Proof.
Proof. Let $C(z_0, R)$ be the circle around $z_0$ of radius $R$, oriented positively.
**Proof.** Let \( C(z_0, R) \) be the circle around \( z_0 \) of radius \( R \), oriented positively.

\[
f(z) = \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{\xi - z} \, d\xi
\]
**Proof.** Let $C(z_0, R)$ be the circle around $z_0$ of radius $R$, oriented positively.

\[
f(z) = \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{\xi - z} \, d\xi = \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{\xi - z_0 - (z - z_0)} \, d\xi
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Proof. Let $C(z_0,R)$ be the circle around $z_0$ of radius $R$, oriented positively.

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\]

\[
 = \frac{1}{2\pi i} \int_{C(z_0,R)} \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} d\xi
\]
Proof. Let $C(z_0, R)$ be the circle around $z_0$ of radius $R$, oriented positively.

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$$= \frac{1}{2\pi i} \int_{C(z_0,R)} \frac{f(\xi)}{\xi - z_0} \left(1 - \frac{z - z_0}{\xi - z_0}\right) \, d\xi$$

$$= \frac{1}{2\pi i} \int_{C(z_0,R)} \frac{f(\xi)}{\xi - z_0} \left[\sum_{n=0}^{N} \left(\frac{z - z_0}{\xi - z_0}\right)^n\right]$$
Proof. Let $C(z_0, R)$ be the circle around $z_0$ of radius $R$, oriented positively.

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= \frac{1}{2\pi i} \oint_{C(z_0, R)} \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \, d\xi \\
= \frac{1}{2\pi i} \oint_{C(z_0, R)} \frac{f(\xi)}{\xi - z_0} \left[ \sum_{n=0}^{N} \left( \frac{z - z_0}{\xi - z_0} \right)^n + \sum_{n=N+1}^{\infty} \left( \frac{z - z_0}{\xi - z_0} \right)^n \right] \, d\xi
\]
**Proof.** Let $C(z_0, R)$ be the circle around $z_0$ of radius $R$, oriented positively.

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f(z) = \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{\xi - z} \, d\xi = \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{\xi - z_0 - (z - z_0)} \, d\xi
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\]

\[
= \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{\xi - z_0} \left[ \sum_{n=0}^{N} \left( \frac{z - z_0}{\xi - z_0} \right)^n + \sum_{n=N+1}^{\infty} \left( \frac{z - z_0}{\xi - z_0} \right)^n \right] \, d\xi
\]

\[
= \sum_{n=0}^{N} (z - z_0)^n \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi
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\]

\[
= \sum_{n=0}^{N} (z - z_0)^n \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi
\]

\[
+ \frac{1}{2\pi i} \int_{C(z_0, R)} \frac{f(\xi)}{\xi - z_0} \left( \frac{z - z_0}{\xi - z_0} \right)^{N+1} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \, d\xi
\]
Proof.
Proof. Now note that, because $z$ is fixed and $|z - z_0| < R$
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\[
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**Proof.** Hence

\[
f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{C(z_0,R)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi
\]

\[= f^{(n)}(z_0) \frac{1}{n!} = a_n\]
Proof. Hence

\[ f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{C(z_0,R)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \ d\xi \]

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Example. *For all complex numbers* $z$ *we have* $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
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Recall that for \( f(z) = e^z \) we have \( f'(z) = e^z \) and hence \( f^{(n)}(z) = e^z \) for all \( z \) and all \( n \).
Example. For all complex numbers $z$ we have $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Recall that for $f(z) = e^z$ we have $f'(z) = e^z$ and hence $f^{(n)}(z) = e^z$ for all $z$ and all $n$. In particular, $f^{(n)}(0) = e^0 = 1$ for all $n$. 

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Series Expansion of Analytic Functions
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Example.

For all complex numbers $z$ we have

$$
\cos(z) = \sum_{n=0}^{\infty} \left(-1\right)^n \frac{z^{2n}}{(2n)!}
$$

Once more the expansion is about the origin $z=0$ and because $f(z) = \cos(z)$ is analytic on any disk around the origin, whatever expansion we find will be valid for all complex numbers $z$.

Note that $f(z) = \cos(z)$, $f'(z) = -\sin(z)$, $f''(z) = -\cos(z)$, $f'''(z) = \sin(z)$, $f''''(z) = \cos(z)$, and after that, it repeats. Thus $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$, $f''''(0) = 1$, and after that, it repeats.

From this pattern, we see that the odd numbered terms $a_{2k+1}$ are all zero. The even numbered terms can be abbreviated as $a_{2n}$. They are alternatingly positive and negative, which can be encoded with $\left(-1\right)^n$.

Thus $a_{2n} = \left(-1\right)^n \frac{z^{2n}}{(2n)!}$. 
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\[ f(z) = \cos(z), \quad f'(z) = -\sin(z), \quad f''(z) = -\cos(z), \quad f'''(z) = \sin(z), \quad f''''(z) = \cos(z) \]
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Example.

For all complex numbers $z$ with $|z| < 1$ we have

$$1 - z = \sum_{n=0}^{\infty} z^n$$

See earlier example!
Example. For all complex numbers $z$ with $|z| < 1$ we have

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\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n
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Example.

\[ f(z) = \frac{1}{z} + z^4 \text{ for } |z| < 1. \]

We will formally deal with negative exponents in these expansions soon.
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\[
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= \frac{1}{z} \cdot \frac{1}{1 - (-z^3)} \\
= \frac{1}{z} \sum_{n=0}^{\infty} (-z^3)^n \\
= \sum_{n=0}^{\infty} \frac{(-1)^n z^{3n}}{z} \\
= \sum_{n=0}^{\infty} (-1)^n z^{3n-1}
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