Derivatives of Complex Functions

Bernd Schröder
Introduction
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4. After all, the algebra and the idea of a limit translate to $\mathbb{C}$. 
Definition.
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It can be proved that $f$ is differentiable if and only if
\[
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\] exists.
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exists. In this case, the above limit is the derivative.

With $\Delta w := f(z_0 + \Delta z) - f(z_0)$ we also write $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$. 
Example.

Compute the derivative of
\[ f(z) = z^3. \]

Differentiation Formulas
Example. Compute the derivative of $f(z) = z^3$. 
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\[
\frac{dz^3}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^3 - z^3}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^3 + 3z^2 \Delta z + 3z \Delta^2 z + \Delta^3 z - z^3}{\Delta z} = \lim_{\Delta z \to 0} \frac{3z^2 \Delta z + 3z \Delta^2 z + \Delta^3 z}{\Delta z} = \lim_{\Delta z \to 0} (3z^2 + 3z \Delta z + \Delta^2 z) = 3z^2.
\]
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\]

... which does not exist.
Theorem.
**Theorem.** *If the complex function* $f$ *is differentiable at* $z_0$, *then* $f$ *is continuous at* $z_0$. 
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$$0 = \lim_{z \to z_0} (z - z_0) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
**Theorem.** *If the complex function* $f$ *is differentiable at* $z_0$, *then* $f$ *is continuous at* $z_0$. *Hence, every differentiable function is continuous. However, not every continuous function is differentiable.*

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For the last statement, consider $f(z) = \bar{z}$. 
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\[\blacksquare\]
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Theorem. Let \( f \) and \( g \) be differentiable at \( z_0 \) and let \( c \in \mathbb{C} \). Then the functions \( f + g \), \( f - g \) and \( cf \) are all differentiable at \( z_0 \) and

\[
(f + g)'(z_0) = f'(z_0) + g'(z_0),
\]

\[
(f - g)'(z_0) = f'(z_0) - g'(z_0),
\]

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(cf)'(z_0) = cf'(z_0).
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Proof (addition only).
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\[(f + g)'(z_0)\]
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\[(f + g)'(z_0)\]

\[= \lim_{z \to z_0} \frac{(f + g)(z) - (f + g)(z_0)}{z - z_0}\]
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\[(f + g)'(z_0)\]

\[= \lim_{z \to z_0} \frac{(f + g)(z) - (f + g)(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) + g(z) - f(z_0) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0) + g(z) - g(z_0)}{z - z_0} = f'(z_0) + g'(z_0)\]
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\[= \lim_{z \to z_0} \frac{f(z) - f(z_0) + g(z) - g(z_0)}{z - z_0} \]

\[= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} + \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} \]
Proof (addition only).

\[(f + g)'(z_0)\]

\[= \lim_{z \to z_0} \frac{(f + g)(z) - (f + g)(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) + g(z) - f(z_0) - g(z_0)}{z - z_0}\]

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\[= f'(z_0) + g'(z_0)\]
**Proof (addition only).**

\[ (f + g)'(z_0) \]

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\[ = f'(z_0) + g'(z_0) \]

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*Louisiana Tech University, College of Engineering and Science*

**Derivatives of Complex Functions**
Theorem.
Theorem. Product and Quotient Rule. Let $f$ and $g$ be differentiable at $z_0$. 

Moreover, if $g(z_0) \neq 0$, then the quotient $\frac{f}{g}$ is differentiable at $z_0$ with 

$$
\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}.
$$
Theorem. Product and Quotient Rule. Let \( f \) and \( g \) be differentiable at \( z_0 \). Then \( fg \) is differentiable at \( x \)
Theorem. Product and Quotient Rule. Let \( f \) and \( g \) be differentiable at \( z_0 \). Then \( fg \) is differentiable at \( x \) with

\[
(fg)'(z_0) = f'(z_0)g(z_0) + g'(z_0)f(z_0).
\]
**Theorem. Product and Quotient Rule.** Let $f$ and $g$ be differentiable at $z_0$. Then $fg$ is differentiable at $x$ with

$$(fg)'(z_0) = f'(z_0)g(z_0) + g'(z_0)f(z_0).$$

Moreover, if $g(z_0) \neq 0$, then the quotient $\frac{f}{g}$ is differentiable at $z_0$. 
Theorem. **Product and Quotient Rule.** Let \( f \) and \( g \) be differentiable at \( z_0 \). Then \( fg \) is differentiable at \( x \) with

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(fg)'(z_0) = f'(z_0)g(z_0) + g'(z_0)f(z_0).
\]

Moreover, if \( g(z_0) \neq 0 \), then the quotient \( \frac{f}{g} \) is differentiable at \( z_0 \) with

\[
\left( \frac{f}{g} \right)'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{(g(z_0))^2}.
\]
Proof (quotient rule only).
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\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \frac{f(z)}{g(z)} - \frac{f(z_0)}{g(z_0)} \]

\[
= \lim_{z \to z_0} \frac{1}{g(z)} \frac{f(z) - f(z_0)}{z - z_0}
\]
Proof (quotient rule only).

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \frac{f(z)}{g(z)} - \lim_{z \to z_0} \frac{f(z_0)}{g(z_0)}
\]
Proof (quotient rule only).

\[
\lim_{z \to z_0} \frac{\frac{f}{g}(z) - \frac{f}{g}(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\frac{f(z)}{g(z)} - \frac{f(z_0)}{g(z_0)}}{z - z_0} = \lim_{z \to z_0} \frac{\frac{f(z)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)}}{z - z_0}
\]
Proof (quotient rule only).

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z)} \cdot \frac{g(z)}{g(z_0)} - \frac{f(z_0)g(z) - f(z)g(z_0)}{g(z)g(z_0)} = \lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( f(z)g(z_0) - f(z_0)g(z) \right)
\]
Proof (quotient rule only).

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)} \cdot \frac{g(z)g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)g(z_0) - f(z_0)g(z) + f(z_0)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)} = \lim_{z \to z_0} \frac{f(z)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)} \cdot \frac{g(z)g(z_0)}{z - z_0}.
\]
Proof (quotient rule only).

\[
\lim_{{z \to z_0}} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{{z \to z_0}} \frac{f(z)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)}
\]

\[
= \lim_{{z \to z_0}} \frac{f(z)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)}
\]

\[
= \lim_{{z \to z_0}} \frac{f(z)g(z_0) - f(z_0)g(z_0) + f(z_0)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)}
\]

\[
= \lim_{{z \to z_0}} \frac{f(z)g(z_0) - f(z_0)g(z_0) - (f(z_0)g(z) - f(z_0)g(z_0))}{{z - z_0}}
\]
Proof (quotient rule only).

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \cdot \frac{g(z) g(z_0)}{g(z) g(z_0)} = \lim_{z \to z_0} \frac{f(z) g(z_0) - f(z_0) g(z)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) g(z_0) - f(z_0) g(z) + f(z_0) g(z_0) - f(z_0) g(z)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) g(z_0) - f(z_0) g(z_0) - f(z_0) g(z_0) + f(z_0) g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) g(z_0) - f(z_0) g(z_0) - f(z_0) g(z_0) + f(z_0) g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) g(z_0) - f(z_0) g(z_0)}{z - z_0} - \frac{f(z_0) g(z) - f(z_0) g(z_0)}{z - z_0}
\]
Proof (quotient rule cont.).
Proof (quotient rule cont.).

\[
\lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z)g(z_0) - f(z_0)g(z)}{z - z_0} - \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0} \right)
\]
Proof (quotient rule cont.).

\[
\lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0} - \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0} \right)
\]

\[= \lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z) - f(z_0)}{z - z_0} g(z_0) - \frac{g(z) - g(z_0)}{z - z_0} f(z_0) \right)\]
Proof (quotient rule cont.).

\[
\lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0} - \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0} \right)
\]

\[
= \lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z) - f(z_0)}{z - z_0} g(z_0) - \frac{g(z) - g(z_0)}{z - z_0} f(z_0) \right)
\]

\[
= \frac{1}{(g(z_0))^2} \left( f'(z_0)g(z_0) - g'(z_0)f(z_0) \right)
\]
Proof (quotient rule cont.).

$$
\lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0} \right)
$$

\[= \lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z) - f(z_0)}{z - z_0} g(z_0) - \frac{g(z) - g(z_0)}{z - z_0} f(z_0) \right) \]

\[= \frac{1}{(g(z_0))^2} \left( f'(z_0)g(z_0) - g'(z_0)f(z_0) \right) \]

\[= \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{(g(z_0))^2}. \]
Proof (quotient rule cont.).

\[
\lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z)g(z_0) - f(z_0)g(z)}{z - z_0} - \frac{f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0} \right)
\]

\[
= \lim_{z \to z_0} \frac{1}{g(z)g(z_0)} \left( \frac{f(z) - f(z_0)}{z - z_0} g(z_0) - \frac{g(z) - g(z_0)}{z - z_0} f(z_0) \right)
\]

\[
= \frac{1}{(g(z_0))^2} \left( f'(z_0)g(z_0) - g'(z_0)f(z_0) \right)
\]

\[
= \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{(g(z_0))^2}.
\]
Theorem.
Theorem. Chain Rule.
Theorem. Chain Rule. Let $f, g$ be complex functions and let $z_0$ be such that $g$ is differentiable at $z_0$ and $f$ is differentiable at $g(z_0)$. 

then $f \circ g$ is differentiable at $z_0$ and the derivative is $$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$
**Theorem. Chain Rule.** Let $f, g$ be complex functions and let $z_0$ be such that $g$ is differentiable at $z_0$ and $f$ is differentiable at $g(z_0)$. Then $f \circ g$ is differentiable at $z_0$. 
Theorem. Chain Rule. Let $f, g$ be complex functions and let $z_0$ be such that $g$ is differentiable at $z_0$ and $f$ is differentiable at $g(z_0)$. Then $f \circ g$ is differentiable at $z_0$ and the derivative is

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$
Proof (small technicality ignored).
Proof (small technicality ignored).

\[
\lim_{z \to z_0} \frac{f \circ g(z) - f \circ g(z_0)}{z - z_0}
\]
Proof (small technicality ignored).

\[
\lim_{z \to z_0} \frac{f \circ g(z) - f \circ g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0} = f'(g(z_0)) g'(z_0).
\]
Proof (small technicality ignored).

\[
\lim_{z \to z_0} \frac{f \circ g(z) - f \circ g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0}
\]
Proof (small technicality ignored).

\[
\lim_{z \to z_0} \frac{f \circ g(z) - f \circ g(z_0)}{z - z_0} =
\lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0}
\]

\[
= \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0}
\]

\[
= \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}
\]

\[
f'(g(z_0)) \cdot g'(z_0).
\]
Proof (small technicality ignored).

\[
\lim_{z \to z_0} \frac{f \circ g(z) - f \circ g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{z - z_0} \cdot \frac{g(z) - g(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \lim_{z \to z_0} \frac{f(u) - f(g(z_0))}{u - g(z_0)} \cdot \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \frac{f'(g(z_0))}{g'(z_0)}.
\]
Proof (small technicality ignored).

\[
\lim_{{z \to z_0}} \frac{f \circ g(z) - f \circ g(z_0)}{z - z_0} = \lim_{{z \to z_0}} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0}
\]

\[
= \lim_{{z \to z_0}} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \lim_{{z \to z_0}} \frac{g(z) - g(z_0)}{z - z_0}
\]

\[
= \lim_{{u \to g(z_0)}} \frac{f(u) - f(g(z_0))}{u - g(z_0)} \cdot \lim_{{z \to z_0}} \frac{g(z) - g(z_0)}{z - z_0}
\]

\[
= f'(g(z_0)) g'(z_0).
\]
Proof (small technicality ignored).

\[
\lim_{z \to z_0} \frac{f \circ g(z) - f \circ g(z_0)}{z - z_0} = \\
= \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0} = \\
= \lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \\
= \lim_{u \to g(z_0)} \frac{f(u) - f(g(z_0))}{u - g(z_0)} \cdot \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \\
= f'(g(z_0)) g'(z_0). \]
Example

Find the derivative of \( f(x) = (iz + 4)^3(z^2 + 1)^2 \).

\[
d_{dz}(iz + 4)^3(z^2 + 1)^2 = 3(iz + 4)^2i(z^2 + 1)^2 + 4(z + 4)^3(2z)(z^2 + 1)^2 = (iz + 4)^2(z^2 + 1)(3iz^2 + 3i + 4iz^2 + 16z)
\]
Example (derivatives work just like they did in calculus).
Example (derivatives work just like they did in calculus). Find the derivative of $f(x) = (iz + 4)^3 (z^2 + 1)^2$.
Example (derivatives work just like they did in calculus). Find the derivative of $f(x) = (iz + 4)^3 (z^2 + 1)^2$.

$$\frac{d}{dz} (iz + 4)^3 (z^2 + 1)^2$$
Example (derivatives work just like they did in calculus). Find the derivative of \( f(x) = (iz + 4)^3 \left(z^2 + 1\right)^2 \).

\[
\frac{d}{dz} \left(iz + 4\right)^3 \left(z^2 + 1\right)^2 = 3 (iz + 4)^2 i \left(z^2 + 1\right)^2
\]
Example (derivatives work just like they did in calculus). Find the derivative of \( f(x) = (iz + 4)^3 (z^2 + 1)^2 \).

\[
\frac{d}{dz} (iz + 4)^3 (z^2 + 1)^2

= 3 (iz + 4)^2 i (z^2 + 1)^2 + (iz + 4)^3 2 (z^2 + 1) 2z
\]
Example (derivatives work just like they did in calculus). Find the derivative of \( f(x) = (iz + 4)^3 (z^2 + 1)^2 \).

\[
\frac{d}{dz} (iz + 4)^3 (z^2 + 1)^2 \\
= 3 (iz + 4)^2 i (z^2 + 1)^2 + (iz + 4)^3 2 (z^2 + 1) 2z \\
= 3i (iz + 4)^2 (z^2 + 1)^2 + 4z (iz + 4)^3 (z^2 + 1)
\]
Example (derivatives work just like they did in calculus). Find the derivative of $f(x) = (iz + 4)^3 (z^2 + 1)^2$.

\[
\frac{d}{dz} (iz + 4)^3 (z^2 + 1)^2
\]

\[
= 3(iz + 4)^2 i (z^2 + 1)^2 + (iz + 4)^3 2 (z^2 + 1) 2z
\]

\[
= 3i(iz + 4)^2 (z^2 + 1)^2 + 4z (iz + 4)^3 (z^2 + 1)
\]

\[
= (iz + 4)^2 (z^2 + 1) (3iz^2 + 3i + 4iz^2 + 16z)
\]
Example (derivatives work just like they did in calculus). Find the derivative of \( f(x) = (iz + 4)^3 (z^2 + 1)^2 \).

\[
\frac{d}{dz} (iz + 4)^3 (z^2 + 1)^2 = 3 (iz + 4)^2 i (z^2 + 1)^2 + (iz + 4)^3 2 (z^2 + 1) 2z \\
= 3i (iz + 4)^2 (z^2 + 1)^2 + 4z (iz + 4)^3 (z^2 + 1) \\
= (iz + 4)^2 (z^2 + 1) (3iz^2 + 3i + 4iz^2 + 16z) \\
= (iz + 4)^2 (z^2 + 1) (7iz^2 + 16z + 3i)
\]