Ordinal Numbers and the Axiom of Substitution

Bernd Schröder
Introduction

1. We spent significant effort to extend $\mathbb{N}$ from the counting system that the Peano Axioms provide to a framework that accommodates algebra and analysis.

2. But we also would like to extend the idea of counting past infinity.

3. “She can’t do that.”

4. Well-ordered sets provide such a mechanism.

5. But we also want to have standardized numbers.

6. This is where ordinal numbers come in.
Introduction

1. We spent significant effort to extend \( \mathbb{N} \) from the counting system that the Peano Axioms provide to a framework that accommodates algebra and analysis.
Introduction

1. We spent significant effort to extend $\mathbb{N}$ from the counting system that the Peano Axioms provide to a framework that accommodates algebra and analysis.

2. But we also would like to extend the idea of counting past infinity.
Introduction

1. We spent significant effort to extend $\mathbb{N}$ from the counting system that the Peano Axioms provide to a framework that accommodates algebra and analysis.
2. But we also would like to extend the idea of counting past infinity.
3. “She can’t do that.”
Introduction

1. We spent significant effort to extend \( \mathbb{N} \) from the counting system that the Peano Axioms provide to a framework that accommodates algebra and analysis.
2. But we also would like to extend the idea of counting past infinity.
3. “She can’t do that.”
4. Well-ordered sets provide such a mechanism.
Introduction

1. We spent significant effort to extend $\mathbb{N}$ from the counting system that the Peano Axioms provide to a framework that accommodates algebra and analysis.

2. But we also would like to extend the idea of counting past infinity.

3. “She can’t do that.”

4. Well-ordered sets provide such a mechanism.

5. But we also want to have standardized numbers.
Introduction

1. We spent significant effort to extend \( \mathbb{N} \) from the counting system that the Peano Axioms provide to a framework that accommodates algebra and analysis.

2. But we also would like to extend the idea of counting past infinity.

3. “She can’t do that.”

4. Well-ordered sets provide such a mechanism.

5. But we also want to have standardized numbers.

6. This is where ordinal numbers come in.
Introduction

Every natural number contains all the natural numbers before it as elements and as strict subsets.
Introduction

\[ 0 = \emptyset \]
Introduction

\[
\begin{align*}
0 &= \emptyset \\
1 &= \{\emptyset\} = \{0\}
\end{align*}
\]
Introduction

\[
\begin{align*}
0 & = \emptyset \\
1 & = \{\emptyset\} = \{0\} \\
2 & = \{\emptyset, \{\emptyset\}\} = \{0, 1\}
\end{align*}
\]
Introduction

\[
\begin{align*}
0 & = \emptyset \\
1 & = \{\emptyset\} = \{0\} \\
2 & = \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 & = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}
\end{align*}
\]
Introduction

\[
\begin{align*}
0 & = \emptyset \\
1 & = \{\emptyset\} = \{0\} \\
2 & = \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 & = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
\vdots
\end{align*}
\]
Introduction

\[
\begin{align*}
0 &= \emptyset \\
1 &= \{\emptyset\} = \{0\} \\
2 &= \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
&\vdots
\end{align*}
\]

Every natural number contains all the natural numbers before it as elements \textit{and} as strict subsets.
Proposition.
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. 
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$. 
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{ k \in n : k \subset m \}$.

**Proof.**
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

**Proof.** Induction on $n$. 
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{ k \in n : k \subset m \}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

*Induction step $n \rightarrow n' = n \cup \{n\}:*
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

*Induction step* $n \rightarrow n' = n \cup \{n\}$: Let $m \in n'$. 
Proposition. Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

Proof. Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

Induction step $n \rightarrow n' = n \cup \{n\}$: Let $m \in n'$. If $m \in n$, then $m = \{k \in n : k \subset m\}$.
Proposition. Consider \( \mathbb{N}_0 \), where \( \mathbb{N} \) is constructed as for the Peano Axioms and \( 0 := \emptyset \). Then every \( n \in \mathbb{N}_0 \) is so that for all \( m \in n \) we have \( m = \{ k \in n : k \subset m \} \).

Proof. Induction on \( n \). The base step \( n = 0 \) is trivial, because \( 0 = \emptyset \) has no elements.

Induction step \( n \to n' = n \cup \{n\} \): Let \( m \in n' \). If \( m \in n \), then \( m = \{ k \in n : k \subset m \} \). Because \( m \subseteq n \), we have \( n \not\subset m \).
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

*Induction step* $n \rightarrow n' = n \cup \{n\}$: Let $m \in n'$. If $m \in n$, then $m = \{k \in n : k \subset m\}$. Because $m \subseteq n$, we have $n \not\subset m$ and so, because $n' \setminus n = \{n\}$, the latter set is equal to $\{k \in n' : k \subset m\}$. 

---

*Bernd Schröder*  
Louisiana Tech University, College of Engineering and Science  
*Ordinal Numbers and the Axiom of Substitution*
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

*Induction step $n \rightarrow n' = n \cup \{n\}$:* Let $m \in n'$. If $m \in n$, then $m = \{k \in n : k \subset m\}$. Because $m \subseteq n$, we have $n \not\subset m$ and so, because $n' \setminus n = \{n\}$, the latter set is equal to $\{k \in n' : k \subset m\}$. If $m \not\in n$, then $m = n$
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{ k \in n : k \subseteq m \}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

*Induction step $n \rightarrow n' = n \cup \{ n \}$:* Let $m \in n'$. If $m \in n$, then $m = \{ k \in n : k \subset m \}$. Because $m \subseteq n$, we have $n \not\subset m$ and so, because $n' \setminus n = \{ n \}$, the latter set is equal to $\{ k \in n' : k \subset m \}$.

If $m \not\in n$, then $m = n = \{ k \in n : k \subseteq n \}$.
Proposition. Consider \( \mathbb{N}_0 \), where \( \mathbb{N} \) is constructed as for the Peano Axioms and \( 0 := \emptyset \). Then every \( n \in \mathbb{N}_0 \) is so that for all \( m \in n \) we have \( m = \{ k \in n : k \subset m \} \).

Proof. Induction on \( n \). The base step \( n = 0 \) is trivial, because \( 0 = \emptyset \) has no elements.

Induction step \( n \to n' = n \cup \{ n \} \): Let \( m \in n' \). If \( m \in n \), then \( m = \{ k \in n : k \subset m \} \). Because \( m \subseteq n \), we have \( n \not\subset m \) and so, because \( n' \setminus n = \{ n \} \), the latter set is equal to \( \{ k \in n' : k \subset m \} \).

If \( m \not\in n \), then \( m = n = \{ k \in n : k \subseteq n \} \) (proved in derivation of the Peano Axioms, and it trivially holds for 0).
Proposition. Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

Proof. Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

Induction step $n \rightarrow n' = n \cup \{n\}$: Let $m \in n'$. If $m \in n$, then $m = \{k \in n : k \subset m\}$. Because $m \subseteq n$, we have $n \not\subset m$ and so, because $n' \setminus n = \{n\}$, the latter set is equal to $\{k \in n' : k \subset m\}$.

If $m \not\in n$, then $m = n = \{k \in n : k \subseteq n\}$ (proved in derivation of the Peano Axioms, and it trivially holds for $0$). Moreover, (good exercise) the containment can be sharpened to strict containment.

Bernd Schröder
Louisiana Tech University, College of Engineering and Science

Ordinal Numbers and the Axiom of Substitution
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements. 

*Induction step* $n \rightarrow n' = n \cup \{n\}$: Let $m \in n'$. If $m \in n$, then $m = \{k \in n : k \subset m\}$. Because $m \subseteq n$, we have $n \not\subset m$ and so, because $n' \setminus n = \{n\}$, the latter set is equal to $\{k \in n' : k \subset m\}$. If $m \not\in n$, then $m = n = \{k \in n : k \subseteq n\}$ (proved in derivation of the Peano Axioms, and it trivially holds for 0). Moreover, (good exercise) the containment can be sharpened to strict containment, so $m = n = \{k \in n : k \subset n\}$.
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{ k \in n : k \subset m \}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

*Induction step* $n \rightarrow n' = n \cup \{n\}$: Let $m \in n'$. If $m \in n$, then $m = \{ k \in n : k \subset m \}$. Because $m \subseteq n$, we have $n \not\subset m$ and so, because $n' \setminus n = \{n\}$, the latter set is equal to $\{ k \in n' : k \subset m \}$. If $m \not\in n$, then $m = n = \{ k \in n : k \subseteq n \}$ (proved in derivation of the Peano Axioms, and it trivially holds for $0$). Moreover, (good exercise) the containment can be sharpened to strict containment, so $m = n = \{ k \in n : k \subset n \}$. Finally, because $n' \setminus n = \{n\}$ and because $n$ is not a strict subset of itself, $m = n = \{ k \in n' : k \subset n \}$.
**Proposition.** Consider $\mathbb{N}_0$, where $\mathbb{N}$ is constructed as for the Peano Axioms and $0 := \emptyset$. Then every $n \in \mathbb{N}_0$ is so that for all $m \in n$ we have $m = \{k \in n : k \subset m\}$.

**Proof.** Induction on $n$. The base step $n = 0$ is trivial, because $0 = \emptyset$ has no elements.

*Induction step* $n \to n' = n \cup \{n\}$: Let $m \in n'$. If $m \in n$, then $m = \{k \in n : k \subset m\}$. Because $m \subseteq n$, we have $n \not\subset m$ and so, because $n' \setminus n = \{n\}$, the latter set is equal to $\{k \in n' : k \subset m\}$. If $m \not\in n$, then $m = n = \{k \in n : k \subseteq n\}$ (proved in derivation of the Peano Axioms, and it trivially holds for 0). Moreover, (good exercise) the containment can be sharpened to strict containment, so $m = n = \{k \in n : k \subset n\}$. Finally, because $n' \setminus n = \{n\}$ and because $n$ is not a strict subset of itself, $m = n = \{k \in n' : k \subset n\}$. 

\[\square\]
Definition.
Definition. An ordinal number is a set $\alpha$ of sets that is well-ordered by set inclusion so that for each $\beta \in \alpha$ we have that $\beta = \{\gamma \in \alpha : \gamma \subset \beta\}$. 
Definition. An ordinal number is a set $\alpha$ of sets that is well-ordered by set inclusion so that for each $\beta \in \alpha$ we have that $\beta = \{\gamma \in \alpha : \gamma \subset \beta\}$.

Proposition.
**Definition.** An **ordinal number** is a set $\alpha$ of sets that is well-ordered by set inclusion so that for each $\beta \in \alpha$ we have that $\beta = \{ \gamma \in \alpha : \gamma \subset \beta \}$.

**Proposition.** Let $\alpha$ be an ordinal number and define the **successor** of $\alpha$ to be $\alpha' := \alpha \cup \{ \alpha \}$.
Definition. An ordinal number is a set $\alpha$ of sets that is well-ordered by set inclusion so that for each $\beta \in \alpha$ we have that $\beta = \{\gamma \in \alpha : \gamma \subset \beta\}$.

Proposition. Let $\alpha$ be an ordinal number and define the successor of $\alpha$ to be $\alpha' := \alpha \cup \{\alpha\}$. Then $\alpha'$ is an ordinal number, too.
**Definition.** An ordinal number is a set $\alpha$ of sets that is well-ordered by set inclusion so that for each $\beta \in \alpha$ we have that $\beta = \{ \gamma \in \alpha : \gamma \subset \beta \}$.

**Proposition.** Let $\alpha$ be an ordinal number and define the successor of $\alpha$ to be $\alpha' := \alpha \cup \{ \alpha \}$. Then $\alpha'$ is an ordinal number, too.

**Proof.**
**Definition.** An ordinal number is a set $\alpha$ of sets that is well-ordered by set inclusion so that for each $\beta \in \alpha$ we have that $\beta = \{ \gamma \in \alpha : \gamma \subset \beta \}$.

**Proposition.** Let $\alpha$ be an ordinal number and define the successor of $\alpha$ to be $\alpha' := \alpha \cup \{ \alpha \}$. Then $\alpha'$ is an ordinal number, too.

**Proof.** Good exercise.
**Definition.** An ordinal number is a set $\alpha$ of sets that is well-ordered by set inclusion so that for each $\beta \in \alpha$ we have that $\beta = \{ \gamma \in \alpha : \gamma \subset \beta \}$.

**Proposition.** Let $\alpha$ be an ordinal number and define the successor of $\alpha$ to be $\alpha' := \alpha \cup \{ \alpha \}$. Then $\alpha'$ is an ordinal number, too.

**Proof.** Good exercise.
Define $\omega + n$ to be the $n$-fold successor of $\omega$
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem).
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem).
Want to define $2\omega$ to be the union of the $\omega + n$
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem).
Want to define $2\omega$ to be the union of the $\omega + n$ (problem: no guarantee for a surrounding set).
Define \( \omega + n \) to be the \( n \)-fold successor of \( \omega \) (no problem).
Want to define \( 2\omega \) to be the union of the \( \omega + n \) (problem: no guarantee for a surrounding set).

**Axiom.**
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem). Want to define $2\omega$ to be the union of the $\omega + n$ (problem: no guarantee for a surrounding set).

**Axiom. Axiom of Substitution.**
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem).
Want to define $2\omega$ to be the union of the $\omega + n$ (problem: no guarantee for a surrounding set).

**Axiom. Axiom of Substitution.** Let $A$ be a set.
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem). Want to define $2\omega$ to be the union of the $\omega + n$ (problem: no guarantee for a surrounding set).

**Axiom. Axiom of Substitution.** Let $A$ be a set. If $S(a, b)$ is a sentence such that for each $a \in A$ the set $\{b : S(a, b)\}$ can be formed.
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem). Want to define $2\omega$ to be the union of the $\omega + n$ (problem: no guarantee for a surrounding set).

**Axiom. Axiom of Substitution.** Let $A$ be a set. If $S(a,b)$ is a sentence such that for each $a \in A$ the set $\{b : S(a,b)\}$ can be formed (“so that a sensible substitute for $a$ can be constructed”)
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem).
Want to define $2\omega$ to be the union of the $\omega + n$ (problem: no guarantee for a surrounding set).

**Axiom. Axiom of Substitution.** Let $A$ be a set. If $S(a,b)$ is a sentence such that for each $a \in A$ the set $\{b : S(a,b)\}$ can be formed ("so that a sensible substitute for $a$ can be constructed"), then there exists a function $F$ with domain $A$ such that $F(a) = \{b : S(a,b)\}$ for all $a \in A$. 
Define $\omega + n$ to be the $n$-fold successor of $\omega$ (no problem).
Want to define $2\omega$ to be the union of the $\omega + n$ (problem: no guarantee for a surrounding set).

**Axiom. Axiom of Substitution.** Let $A$ be a set. If $S(a, b)$ is a sentence such that for each $a \in A$ the set $\{ b : S(a, b) \}$ can be formed ("so that a sensible substitute for $a$ can be constructed"), then there exists a function $F$ with domain $A$ such that $F(a) = \{ b : S(a, b) \}$ for all $a \in A$. (And because $F$ has a range $R$ that is a set, $\{ F(a) \in R : a \in A \}$ is a set, obtained by substituting $F(a)$ for each $a \in A$.)

Bernd Schröder
Louisiana Tech University, College of Engineering and Science

Ordinal Numbers and the Axiom of Substitution
For $2\omega$
For $2\omega$, let $C(x) := x \cup \{x\}$
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. 
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n)$
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n) = \{b : S(n, b)\}$.
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n) = \{b : S(n, b)\} = \{C^n(\omega)\}$.
For \( 2\omega \), let \( C(x) := x \cup \{x\} \), \( A := \omega = \mathbb{N}_0 \) and 
\[ S(n, b) := [b = C^n(\omega)]. \]
Then \( F(n) = \{b : S(n, b)\} = \{C^n(\omega)\} \)
and \( \bigcup \{F(n) : n \in \mathbb{N}_0\} = \{C^n(\omega) : n \in \mathbb{N}_0\} \)
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n) = \{ b : S(n, b) \} = \{ C^n(\omega) \}$ and $\bigcup \{ F(n) : n \in \mathbb{N}_0 \} = \{ C^n(\omega) : n \in \mathbb{N}_0 \}$ contains $\omega + 0 = \omega = C^0(\omega)$.
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n) = \{ b : S(n, b) \} = \{ C^n(\omega) \}$ and $\bigcup \{ F(n) : n \in \mathbb{N}_0 \} = \{ C^n(\omega) : n \in \mathbb{N}_0 \}$ contains $\omega + 0 = \omega = C^0(\omega)$, $\omega + 1 = \omega' = C^1(\omega)$.
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n) = \{ b : S(n, b) \} = \{ C^n(\omega) \}$ and $\bigcup \{ F(n) : n \in \mathbb{N}_0 \} = \{ C^n(\omega) : n \in \mathbb{N}_0 \}$ contains $\omega + 0 = \omega = C^0(\omega)$, $\omega + 1 = \omega' = C^1(\omega)$, ...
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n) = \{ b : S(n, b) \} = \{ C^n(\omega) \}$ and $\bigcup \{ F(n) : n \in \mathbb{N}_0 \} = \{ C^n(\omega) : n \in \mathbb{N}_0 \}$ contains $\omega + 0 = \omega = C^0(\omega)$, $\omega + 1 = \omega' = C^1(\omega)$, $\ldots$, $\omega + n = C^n(\omega)$.
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and $S(n, b) := [b = C^n(\omega)]$. Then $F(n) = \{ b : S(n, b) \} = \{ C^n(\omega) \}$ and $\bigcup \{ F(n) : n \in \mathbb{N}_0 \} = \{ C^n(\omega) : n \in \mathbb{N}_0 \}$ contains $\omega + 0 = \omega = C^0(\omega)$, $\omega + 1 = \omega' = C^1(\omega)$, $\ldots$, $\omega + n = C^n(\omega)$, etc.
For $2\omega$, let $C(x) := x \cup \{x\}$, $A := \omega = \mathbb{N}_0$ and

$S(n,b) := [b = C^n(\omega)]$. Then $F(n) = \{b : S(n,b)\} = \{C^n(\omega)\}$ and

$\bigcup \{F(n) : n \in \mathbb{N}_0\} = \{C^n(\omega) : n \in \mathbb{N}_0\}$ contains

$\omega + 0 = \omega = C^0(\omega)$, $\omega + 1 = \omega' = C^1(\omega)$, \ldots, $\omega + n = C^n(\omega)$, etc. Now $2\omega := \omega \cup \{C^n(\omega) : n \in \mathbb{N}_0\}$ is an ordinal number (good exercise).
Final Remarks
Final Remarks

1. There is no set of all ordinal numbers (Burali-Forti paradox).
Final Remarks

1. There is no set of all ordinal numbers (Burali-Forti paradox).
2. The ordinal numbers are “totally ordered” by inclusion.
Final Remarks

1. There is no set of all ordinal numbers (Burali-Forti paradox).
2. The ordinal numbers are “totally ordered” by inclusion.
3. We won’t go into set theory with classes, etc.