Variables and Quantifiers

Bernd Schröder
Open Sentences

1. We need to talk about unspecified objects in a set.
2. A sentence that includes symbols (called variables, like $x$, $y$, etc., and which becomes a statement when all variables are replaced with objects taken from a given set is called an open sentence.
3. We will have no qualms using our intuition about familiar sets like $\mathbb{N}$, $\mathbb{Z}$, even before we formally define them.
4. The statement "$x \in S$" will denote the fact that $x$ is an element of the set $S$. 
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4. The statement “$x \in S$” will denote the fact that $x$ is an element of the set $S$. 
The Universal Quantifier $\forall$

Definition. Let $p(x)$ be an open sentence that depends on the variable $x$ and let $S$ be a set. The statement $\forall x \in S : p(x)$ (read "for all $x$ in $S$ we have $p(x)$") is true iff $p(x)$ holds for all elements $x$ in the set $S$.

▶ The symbol $\forall$ is the universal quantifier.
▶ The statement $\forall n \in \mathbb{N} : n \geq 1$ is a true universally quantified statement.
▶ The statement $\forall n \in \mathbb{Z} : n \geq 1$ is a false universally quantified statement.
▶ The statement "Every eight foot tall man is a professional basketball player." is a vacuously true universally quantified statement.
The Universal Quantifier \( \forall \)

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- The statement “Every eight foot tall man is a professional basketball player.” is a **vacuously true** universally quantified statement.
Universal Quantifiers and Implications

The statement $\forall x \in S: p(x)$ is true iff the statement "Let $x$ be an object. If $x \in S$, then $p(x)$." is true.

With $p(x) =$ "$x$ is a professional basketball player" and $S$ being the set of eight foot tall men, the vacuously true statement from the previous slide reads as "If $x \in S$, then $p(x)$."
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With p(x) = “x is a professional basketball player” and S being the set of eight foot tall men, the vacuously true statement from the previous slide reads as “If x ∈ S, then p(x)”. Vacuous truth is identified as an implication with false hypothesis.
The Existential Quantifier $\exists$

Definition. Let $p(x)$ be an open sentence that depends on the variable $x$ and let $S$ be a set. The statement $\exists x \in S : p(x)$ (read "there is an $x$ in $S$ so that $p(x)$") is true iff $p(x)$ is true for at least one element $x$ in the set $S$.

▶ The symbol $\exists$ is the existential quantifier.
▶ The statement $\exists n \in \mathbb{N} : n^2 = 4$ is a true existentially quantified statement.
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Nested Quantifications

Let $f$ be a function and let $a \in \mathbb{R}$. The statement "\( \forall \varepsilon > 0 : \exists \delta > 0 : \forall |x - a| < \delta : |f(x) - f(a)| < \varepsilon \)" is a triply nested quantified statement. If we assume that the outer quantifications fix "their" variables, we won't need to formally define multiple quantifications using open sentences with several variables. (Let's assume just that so that we don't over-formalize our language at this stage.) The statement is the formal definition of "The function $f$ is continuous at $a$."

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Nested Quantifications

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# Nested Quantifications

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Negation of Quantified Statements

1. Some open sentences can be true or false depending on what objects (say, functions or points in the statement on the previous slide) we use in them.

2. Stating that a quantified statement is not true usually gives us less insight than rewriting it in more natural language. For example it would be nice to have a formal statement that says "\( f \) is not continuous at \( a \)."

3. \[ \neg (\forall x \in S: p(x)) = \exists x \in S: (\neg p(x)) \]

4. \[ \neg (\exists x \in S: p(x)) = \forall x \in S: (\neg p(x)) \]

5. These two rules are the main reason why working mathematicians use quantifiers as a tool. Ultimately, they should become as automatic as simplification rules in algebra.
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