Russell’s Paradox

Bernd Schröder
If We Could Define Sets, The Following Should Hold
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set.
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.)
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.) We write $x \in A$ if the object $x$ is in the set $A$. 
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.) We write $x \in A$ if the object $x$ is in the set $A$. (Only notation.)
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.) We write $x \in A$ if the object $x$ is in the set $A$. (Only notation.)

3. Sets $A$ can be formed by specifying their elements with open sentences $p(x)$.
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.) We write $x \in A$ if the object $x$ is in the set $A$. (Only notation.)

3. Sets $A$ can be formed by specifying their elements with open sentences $p(x)$. That is, the element $x$ is in the set $A$ iff $p(x)$ is true.
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.) We write $x \in A$ if the object $x$ is in the set $A$. (Only notation.)

3. Sets $A$ can be formed by specifying their elements with open sentences $p(x)$. That is, the element $x$ is in the set $A$ iff $p(x)$ is true. (Indispensable demand.)
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.) We write $x \in A$ if the object $x$ is in the set $A$. (Only notation.)

3. Sets $A$ can be formed by specifying their elements with open sentences $p(x)$. That is, the element $x$ is in the set $A$ iff $p(x)$ is true. (Indispensable demand.) The set of elements for which $p(x)$ holds is also denoted $\{x : p(x)\}$.
If We Could Define Sets, The Following Should Hold

1. Sets and objects have been defined and can be identified using the definition.

2. For any object and any set we can determine if the object is in the set. (Indispensable demand.) We write $x \in A$ if the object $x$ is in the set $A$. (Only notation.)

3. Sets $A$ can be formed by specifying their elements with open sentences $p(x)$. That is, the element $x$ is in the set $A$ iff $p(x)$ is true. (Indispensable demand.) The set of elements for which $p(x)$ holds is also denoted $\{x : p(x)\}$. (Only notation.)
\( B := \{ A : A \text{ is a set } \land A \notin A \} \text{ Is a Set} \)
\[ B := \{ A : A \text{ is a set } \land A \not\in A \} \text{ Is a Set} \]

By 1, \( p(x) = \text{“} x \text{ is a set”} \) is an open sentence.
\[ B := \{ A : A \text{ is a set } \land A \notin A \} \text{ Is a Set} \]

By 1, \( p(x) = \text{“}x\text{ is a set”} \) is an open sentence.
By 2, \( q(x) = \text{“}x \notin x\text{”} \) is an open sentence.
\[ B := \{ A : A \text{ is a set } \land A \notin A \} \text{ Is a Set} \]

By 1, \( p(x) = "x \text{ is a set}" \) is an open sentence.
By 2, \( q(x) = "x \notin x" \) is an open sentence.
Hence \( r(x) := p(x) \land q(x) \) is an open sentence
\( \mathcal{B} := \{ A : A \text{ is a set } \land A \notin A \} \) Is a Set

By 1, \( p(x) = \text{“}x \text{ is a set”} \) is an open sentence.

By 2, \( q(x) = \text{“}x \notin x \text{”} \) is an open sentence.

Hence \( r(x) := p(x) \land q(x) \) is an open sentence,

and by 3, \( \mathcal{B} = \{ A : r(A) \} \) is a set.
Properties Sets Should Have

The Problem

The Remedy

But Investigating $B = \{A : A \text{ is a set } \land A \notin A\}$ Leads To a Contradiction
But Investigating $\mathcal{B} = \{A : A \text{ is a set } \land A \notin A\}$

Leads To a Contradiction

The statement $\mathcal{B} \notin \mathcal{B}$ is either true or false.
But Investigating $\mathcal{B} = \{A : A \text{ is a set } \land A \not\in A\}$

Leads To a Contradiction

The statement $\mathcal{B} \not\in \mathcal{B}$ is either true or false.
If $\mathcal{B} \not\in \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain
that $\mathcal{B}$ is an element of $\mathcal{B}$.
But Investigating $\mathcal{B} = \{A : A \text{ is a set } \land A \notin A\}$

Leads To a Contradiction

The statement $\mathcal{B} \notin \mathcal{B}$ is either true or false.

If $\mathcal{B} \notin \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain that $\mathcal{B}$ is an element of $\mathcal{B}$. This contradicts our assumption $\mathcal{B} \notin \mathcal{B}$.

If $\mathcal{B} \notin \mathcal{B}$ is false, then $\mathcal{B} \in \mathcal{B}$.

Hence $\mathcal{B}$ satisfies the condition "$A \notin A$" that is required for membership in $\mathcal{B}$.

This is a contradiction, because we have derived $\mathcal{B} \notin \mathcal{B}$ (just done) and $\mathcal{B} \in \mathcal{B}$ (previous paragraph).

So one of 1-3 must be abandoned, because assuming that all three are true leads to a contradiction.
But Investigating $\mathcal{B} = \{A : A \text{ is a set } \land A \notin A\}$

Leads To a Contradiction

The statement $\mathcal{B} \notin \mathcal{B}$ is either true or false.

If $\mathcal{B} \notin \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain that $\mathcal{B}$ is an element of $\mathcal{B}$. This contradicts our assumption $\mathcal{B} \notin \mathcal{B}$. Thus $\mathcal{B} \notin \mathcal{B}$ is false.
But Investigating $B = \{A : A \text{ is a set } \land A \notin A\}$

Leads To a Contradiction

The statement $B \notin B$ is either true or false.
If $B \notin B$ was true, then, by definition of $B$ we would obtain that $B$ is an element of $B$. This contradicts our assumption $B \notin B$. Thus $B \notin B$ is false.
If $B \notin B$ is false, then $B \in B$.
But Investigating $\mathcal{B} = \{A : A \text{ is a set } \land A \notin A\}$

Leads To a Contradiction

The statement $\mathcal{B} \notin \mathcal{B}$ is either true or false.
If $\mathcal{B} \notin \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain that $\mathcal{B}$ is an element of $\mathcal{B}$. This contradicts our assumption $\mathcal{B} \notin \mathcal{B}$. Thus $\mathcal{B} \notin \mathcal{B}$ is false.
If $\mathcal{B} \notin \mathcal{B}$ is false, then $\mathcal{B} \in \mathcal{B}$. Hence $\mathcal{B}$ satisfies the condition “$A \notin A$” that is required for membership in $\mathcal{B}$.
But Investigating $\mathcal{B} = \{A : A \text{ is a set } \land A \not\in A\}$

Leads To a Contradiction

The statement $\mathcal{B} \not\in \mathcal{B}$ is either true or false.

If $\mathcal{B} \not\in \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain that $\mathcal{B}$ is an element of $\mathcal{B}$. This contradicts our assumption $\mathcal{B} \not\in \mathcal{B}$. Thus $\mathcal{B} \not\in \mathcal{B}$ is false.

If $\mathcal{B} \not\in \mathcal{B}$ is false, then $\mathcal{B} \in \mathcal{B}$. Hence $\mathcal{B}$ satisfies the condition “$A \not\in A$” that is required for membership in $\mathcal{B}$. Hence $\mathcal{B} \not\in \mathcal{B}$.
But Investigating $\mathcal{B} = \{ A : A \text{ is a set } \land A \not\in A \}$

Leads To a Contradiction

The statement $\mathcal{B} \not\in \mathcal{B}$ is either true or false.
If $\mathcal{B} \notin \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain that $\mathcal{B}$ is an element of $\mathcal{B}$. This contradicts our assumption $\mathcal{B} \notin \mathcal{B}$. Thus $\mathcal{B} \notin \mathcal{B}$ is false.
If $\mathcal{B} \notin \mathcal{B}$ is false, then $\mathcal{B} \in \mathcal{B}$. Hence $\mathcal{B}$ satisfies the condition “$A \notin A$” that is required for membership in $\mathcal{B}$. Hence $\mathcal{B} \notin \mathcal{B}$. This is a contradiction, because we have derived $\mathcal{B} \notin \mathcal{B}$ (just done) and $\mathcal{B} \in \mathcal{B}$ (previous paragraph).
But Investigating $\mathcal{B} = \{ A : A \text{ is a set } \land A \not\in A \}$

Leads To a Contradiction

The statement $\mathcal{B} \not\in \mathcal{B}$ is either true or false.
If $\mathcal{B} \not\in \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain that $\mathcal{B}$ is an element of $\mathcal{B}$. This contradicts our assumption $\mathcal{B} \not\in \mathcal{B}$. Thus $\mathcal{B} \not\in \mathcal{B}$ is false.

If $\mathcal{B} \not\in \mathcal{B}$ is false, then $\mathcal{B} \in \mathcal{B}$. Hence $\mathcal{B}$ satisfies the condition “$A \not\in A$” that is required for membership in $\mathcal{B}$. Hence $\mathcal{B} \not\in \mathcal{B}$.

This is a contradiction, because we have derived $\mathcal{B} \not\in \mathcal{B}$ (just done) and $\mathcal{B} \in \mathcal{B}$ (previous paragraph).

So one of 1-3 must be abandoned.
But Investigating $\mathcal{B} = \{A : A \text{ is a set } \land A \not\in A\}$

Leads To a Contradiction

The statement $\mathcal{B} \not\in \mathcal{B}$ is either true or false.
If $\mathcal{B} \not\in \mathcal{B}$ was true, then, by definition of $\mathcal{B}$ we would obtain that $\mathcal{B}$ is an element of $\mathcal{B}$. This contradicts our assumption $\mathcal{B} \not\in \mathcal{B}$. Thus $\mathcal{B} \not\in \mathcal{B}$ is false.
If $\mathcal{B} \not\in \mathcal{B}$ is false, then $\mathcal{B} \in \mathcal{B}$. Hence $\mathcal{B}$ satisfies the condition “$A \not\in A$” that is required for membership in $\mathcal{B}$. Hence $\mathcal{B} \not\in \mathcal{B}$.
This is a contradiction, because we have derived $\mathcal{B} \not\in \mathcal{B}$ (just done) and $\mathcal{B} \in \mathcal{B}$ (previous paragraph).
So one of 1-3 must be abandoned, because assuming that all three are true leads to a contradiction.
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.

2. Assumption 2 cannot be dropped: We must be able to check if an object is in a set.

3. Assumption 3 cannot be dropped: How else would we form sets?

4. That only leaves us with 1 as expendable. So, no matter how weird it sounds, we cannot define sets and objects.

Bernd Schröder
Louisiana Tech University, College of Engineering and Science
Russell’s Paradox
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.

2. Assumption 2 cannot be dropped.
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.
2. Assumption 2 cannot be dropped: We must be able to check if an object is in a set.
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.
2. Assumption 2 cannot be dropped: We must be able to check if an object is in a set.
3. Assumption 3 cannot be dropped
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.
2. Assumption 2 cannot be dropped: We must be able to check if an object is in a set.
3. Assumption 3 cannot be dropped: How else would we form sets?
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.
2. Assumption 2 cannot be dropped: We must be able to check if an object is in a set.
3. Assumption 3 cannot be dropped: How else would we form sets?
4. That only leaves us with 1 as expendable.
What Can We Drop?

1. We may not like it, but logic dictates that at least one of 1-3 cannot hold.
2. Assumption 2 cannot be dropped: We must be able to check if an object is in a set.
3. Assumption 3 cannot be dropped: How else would we form sets?
4. That only leaves us with 1 as expendable.

So, no matter how weird it sounds, we cannot define sets and objects.
How Bad Is The Problem?

1. Obviously, it does not stop us from doing mathematics.
2. We will be able to picture sets like we are used to doing.
3. But we must build the theory in such a way that self-referential constructions, like the one for $B$ in Russell's Paradox are avoided.
4. Even though it feels like we are limited, it is possible to build a consistent set theory that is rich enough to allow the construction of the real numbers.
How Bad Is The Problem?

1. Obviously, it does not stop us from doing mathematics.
How Bad Is The Problem?

1. Obviously, it does not stop us from doing mathematics.
2. We will be able to picture sets like we are used to doing.
How Bad Is The Problem?

1. Obviously, it does not stop us from doing mathematics.
2. We will be able to picture sets like we are used to doing.
3. But we must build the theory in such a way that self-referential constructions, like the one for $B$ in Russell’s Paradox are avoided.
How Bad Is The Problem?

1. Obviously, it does not stop us from doing mathematics.
2. We will be able to picture sets like we are used to doing.
3. But we must build the theory in such a way that self-referential constructions, like the one for $B$ in Russell’s Paradox are avoided.
4. Even though it feels like we are limited
How Bad Is The Problem?

1. Obviously, it does not stop us from doing mathematics.
2. We will be able to picture sets like we are used to doing.
3. But we must build the theory in such a way that self-referential constructions, like the one for $B$ in Russell’s Paradox are avoided.
4. Even though it feels like we are limited, it is possible to build a consistent set theory that is rich enough to allow the construction of the real numbers.