JOURNAL OF MATHEMATICAL PHYSICS **50**, 102901 (2009)

Collinear central configurations in the n-body problem with general homogeneous potential

Mervin Woodlin and Zhifu Xie[a)]
Department of Mathematics and Computer Science, Virginia State University, Petersburg, Virginia 23806, USA

(Received 11 June 2009; accepted 23 July 2009; published online 16 October 2009)

In this paper we investigate the central configurations of collinear n-body problem given by the general law of attraction of the form $f(r)=1/r^{\alpha}$. A method involving analysis skills of some elementary algebra and calculus is presented to study the central configurations in the collinear n-body problem. It is well known that for given n positive masses, there are precisely $n!/2$ collinear central configurations for Newton's law of gravitation of $\alpha=2$. However, it is not true that there is always a position that causes a central configuration for any given ordered particles with some positive masses and that there may exist more than one position that make it central for some $\alpha<0$. We give a generalization of Moulton's theorem for collinear n-body problem for all $\alpha>0$. Examples that Moulton's theorem does not work are also provided. © *2009 American Institute of Physics.*
[doi:10.1063/1.3205451]

I. INTRODUCTION AND MAIN RESULTS

Consider the n-body problem in the general law of attraction $f(r)$ with the Newton's law of gravitation ($f(r)=1/r^2$) as a special case,

$$m_k\ddot{q}_k=\sum_{j=1,j\neq k}^{n} m_k m_j f(r_{jk})\frac{(q_j-q_k)}{|q_j-q_k|},\quad 1\leq k\leq n, \tag{1.1}$$

where $m_k>0$ are the masses of the bodies, $q_k\in\mathbf{R}^3$ are their positions, and $r_{jk}=|q_j-q_k|$ is the distance of bodies m_j and m_k. Let

$$C=m_1q_1+\cdots+m_nq_n,\quad M=m_1+\cdots+m_n,\quad c=C/M$$

be the first moment, total mass, and center of mass of the bodies, respectively.

Rather than solving this notoriously difficult system of equations, it would be much easier if it could be reduced to

$$m_k\ddot{q}_k=-\lambda(t)(q_k-c),$$

where λ is a scalar function for all particles. Then the motion at any fixed time must satisfy the following nonlinear algebraic system:

[a)]Electronic addresses: zxie@vsu.edu and xiezhifu@hotmail.com.

 © 2009 American Institute of Physics

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

$$\sum_{j=1,j\neq k}^{n} m_j f(r_{jk}) \frac{(q_j - q_k)}{|q_j - q_k|} = -\lambda(q_k - c), \quad 1 \le k \le n \tag{1.2}$$

for a constant λ. Because the system (1.1) is singular for $f(r)=1/r^{\alpha}(\alpha>0)$ when two particles have the same position, it is natural to assume that the configuration avoids the collision set, which is defined by

$$\Delta = \cup \{q=(q_1,q_2,\cdots,q_n) \in (\mathbf{R}^3)^n | q_i = q_j \text{ for some } i \neq j\}. \tag{1.3}$$

Definition 1.1: (Central configuration) A configuration $q=(q_1,q_2,\ldots,q_n)\in \mathbf{R}^{3n}\setminus\Delta$ is a *central configuration* (CC for short) for $m=(m_1,m_2,\ldots,m_n)\in(\mathbf{R}^+)^n$ if q is a solution of the system (1.2) for some constant $\lambda\in\mathbf{R}$. Two configurations q and $p\in(\mathbf{R}^3)^n\setminus\Delta$ are *equivalent*, denoted by $q \sim p$, if and only if q and p differ by a rotation, followed by a scalar multiplication. This defines an equivalent relation among elements in CCs for m. From now on, the number of CCs refers to the number of equivalent classes. q is *linear* if all the q_i's lie on a straight line in $\mathbf{R}^3$.

When $f(r)=1/r^2$ is the Newton's law of gravitation, CCs play a crucial role for understanding the dynamics of n-body problems.[6,10] In particular, they have led to important theoretical investigations which are connected to smale's sixth problem,[11] originally proposed by Wintner in 1941.[13] Euler discovered the collinear configurations for the three-body problem and Moulton[7] analyzed the general N-body case. Moulton[7] and Albouy and Moeckel[1] also studied the inverse problem of collinear CCs where they proved the following for $\alpha=2$.

Theorem 1.2: *(Reference 7) For the n-body problem with positive masses, there are precisely $n!/2$ collinear CCs. More precisely for each way the particles can be ordered along a line, there is a unique position that causes a CC.*

Theorem 1.3: *(References 1 and 7) For any given configuration of three particles on a line, there always exist three positive masses to make the configuration central. Furthermore, the center of mass only depends on the configuration and is independent of the choices of masses.*

When the n-body problem takes the quasihomogeneous form of $\alpha/r^a+\beta/r^b$, Diacu *et al.*[3] and Jones[4] studied the general properties related to CCs. References 5, 8, 9, and 14–16 give more properties of collinear three- and four-body CCs for Newton's case. References 2, 17, and 18 give some properties in some particular type of CCs. In this paper, our main goal is to study the problem of CCs when the law of attraction is in the homogenous form of $f(r)=1/r^{\alpha}$. We first investigate CCs in collinear three-body problem and then we calculate the number of CCs for any collinear n-body problem, which generalizes Theorem 1.2 proved by Moulton[7] in 1910. Our methods are different from those in Refs. 3 and 12 and they mainly involved analysis skills of some elementary algebra and calculus. We have the following main results.

Theorem 1.4: *Suppose the law of attraction is in the homogenous form of $f(r)=1/r^{\alpha}$. For any given configuration of three particles on a line, we show the following:*

(i) *When $\alpha\in(-1,+\infty)$, there always exist three positive masses to make the configuration central. Furthermore, the center of mass only depends on the configuration and is independent of the choices of masses.*
(ii) *When $\alpha=-1$, any three positive masses can make the configuration central. However, the center of mass is not independent of the choice of masses.*
(iii) *When $\alpha\in(-\infty,-1)$, there always exist three positive masses to make the configuration central. Furthermore, the center of mass only depends on the configuration and is independent of the choices of masses.*

Theorem 1.5: *For any $\alpha>0$, $f(r)=1/r^{\alpha}$, and $m=(m_1,m_2,\ldots,m_n)\in(\mathbf{R}^+)^n$, there are exactly $n!/2$ collinear CCs. More precisely for each way that the particles can be ordered along a line, there is a unique position that causes a CC.*

Remark 1.6: For $\alpha<0$, it is not true that there is a position that causes a CC for any given ordered particles with some positive masses. For $\alpha<0$ and an ordered particles with given posi-

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

tive masses, there may exist more than one position that make it central. Examples are given in Remark 3.4.

Throughout the paper, we assume that $f(r)=1/r^{\alpha}$. In Sec. II, we study the inverse problem of CCs in collinear three-body problem and prove Theorem 1.4 and give some examples. In Sec. III, we first prove that there exists exactly one CC for each given positive masses m_1, m_2, and m_3 in a fixed order on the line and then we prove Theorem 1.5 by induction.[7]

II. INVERSE PROBLEM OF CCs IN COLLINEAR THREE-BODY PROBLEM

In this section, we study the inverse problem of CCs in collinear three-body problem. Given a collinear configuration of three bodies, under what conditions it is possible to choose positive masses which make it central. Because CC is invariant up to translation and scaling, we can choose the coordinate system so that all the three bodies are on the positions $q_1=0$, $q_2=1$, and $q_3=1+r$, where $r>0$. Up to translation and scaling, this is the general three-body configuration. Then (1.2) is equivalent to

$$m_2+\frac{m_3}{(1+r)^{\alpha}}-\lambda c=0,$$

$$-m_1+\frac{m_3}{r^{\alpha}}+\lambda(1-c)=0, \tag{2.1}$$

$$-\frac{m_1}{(1+r)^{\alpha}}-\frac{m_2}{r^{\alpha}}+\lambda(1+r-c)=0,$$

where $c=(m_2+m_3(1+r))/M$ is the center of mass. M would be regarded as a parameter. Given any $r>0$, we want to know whether there exist positive masses m_1, m_2, and m_3, which are solutions for (2.1) with appropriate choices of λ, M. Substituting c into the above equation, (2.1) becomes a linear equation of the masses m_1, m_2, and m_3. By row reduction, we have

$$m_1=-\frac{(1+r)^{\alpha}(-M+r^{\alpha+1}\lambda)}{(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}},$$

$$m_2=\frac{r^{\alpha}((1+r)^{\alpha+1}\lambda-M)}{(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}}, \tag{2.2}$$

$$m_3=-\frac{(\lambda-M)(1+r)^{\alpha}r^{\alpha}}{(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}},$$

except for $\alpha=-1$, where the denominator is zero. By direct computation and simplification, we have the total mass M,

$$M=m_1+m_2+m_3,$$

and the center of mass c,

$$c=\frac{m_2+m_3(1+r)}{M}=\frac{r^{\alpha}((1+r)^{\alpha+1}-1)}{(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}},$$

which is independent of the choices of masses.

Case 1 $(-1<\alpha<+\infty)$*:* When $\alpha\geq 0$, the common denominator of (2.2) is $d(r)=(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}=r^{\alpha}((1+r)^{\alpha}-1)+(1+r)^{\alpha}>0$ for $r>0$. When $-1<\alpha<0$, $d(r)=(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}=(1+r)^{\alpha}r^{\alpha}(1+r^{-\alpha}-(1+r)^{-\alpha})>0$. To have positive masses, we need

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

$$\frac{M}{\lambda} > r^{\alpha+1}, \frac{M}{\lambda} < (1+r)^{\alpha+1}, \frac{M}{\lambda} > 1.$$

So it is always possible to choose the parameters so that all three masses are positive and we need only take

$$\max\{1, r^{\alpha+1}\} < \frac{M}{\lambda} < (1+r)^{\alpha+1}. \tag{2.3}$$

Case 2 ($\alpha=-1$)*:* When $\alpha=-1$, any positive masses m_1, m_2, and m_3 would satisfy Eq. (2.1) if we choose $\lambda=M=m_1+m_2+m_3$. The center of mass c is not fixed and depends on the choices of masses for a given position $r>0$.

Case 3 ($-\infty<\alpha<-1$)*:* When $-\infty<\alpha<-1$, $d(r)=(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}=(1+r)^{\alpha}r^{\alpha}(1+r^{-\alpha}-(1+r)^{-\alpha})<0$. It is always possible to choose the parameters so that all three masses are positive and we need only take

$$(1+r)^{\alpha+1} < \frac{M}{\lambda} < \min\{1, r^{\alpha+1}\}. \tag{2.4}$$

III. CCs IN COLLINEAR THREE-BODY PROBLEM

We want to prove that there exists exactly one positive r, which is the unique positive solution of (2.1) with appropriate λ for any given positive masses m_1, m_2, and m_3 when $\alpha>0$. Solving for M and λ from the first two equations, we have

$$\lambda = m_3(1+r)^{-\alpha} - m_3 r^{-\alpha} + m_2 + m_1,$$

$$M = \frac{\lambda(m_2 + m_3(1+r))(1+r)^{\alpha}}{m_2(1+r)^{\alpha} + m_3}. \tag{3.1}$$

Substituting λ, M into the third equation in (2.1), we have

$$m_1(1+r) + m_2 r + m_3(1+r)^{-\alpha} r - m_1(1+r)^{-\alpha} - m_2 r^{-\alpha} - m_3(1+r) r^{-\alpha} = 0.$$

The positive solutions of the above equation for any given positive m_1, m_2, and m_3 do not change by multiplying $(1+r)^{\alpha}r^{\alpha}$ on both sides,

$$P(r,\alpha) = (1+r)^{\alpha+1} r^{\alpha} m_1 + (1+r)^{\alpha} r^{1+\alpha} m_2 + r^{1+\alpha} m_3 - r^{\alpha} m_1 - m_2(1+r)^{\alpha} - (1+r)^{\alpha+1} m_3 = 0. \tag{3.2}$$

For any given $\alpha>0$, each positive r such that $P(r,\alpha)=0$ produces one CC for collinear three-body problem.

Lemma 3.1: Suppose that $P(r,\alpha)$ is defined as in (3.2). Then for any $\alpha>0$, there exists a unique $r=r_0(\alpha)$ such that (i) $P(r,\alpha)<0$ for $0<r<r_0(\alpha)$, (ii) $P(r_0(\alpha),\alpha)=0$, and (iii) $P(r,\alpha)>0$ for $r>r_0(\alpha)$. Moreover, $\Gamma=\{(r_0(\alpha),\alpha):0<\alpha<\infty\}$ is a smooth curve in the first quadrant of the αr-plane.

Proof: $P(r,\alpha)$ is a continuous function on the first quadrant $r>0, \alpha>0$. By direct computation, for any fixed $m_1>0, m_2>0, m_3>0$, we have

$$(1)\ \ P(0,\alpha) = -m_2 - m_3 < 0, \quad (2)\ \ \lim_{r\to+\infty} P(r,\alpha) = +\infty.$$

By the intermediate value theorem, there exists at least one $r\in(0,\infty)$ such that $P(r,\alpha)=0$. We calculate $\partial P/\partial r$,

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

$$\frac{\partial P}{\partial r}(r,\alpha) = (\alpha+1)(1+r)^{\alpha}r^{\alpha}m_1 + \alpha(1+r)^{\alpha+1}r^{\alpha-1}m_1 + \alpha(1+r)^{\alpha-1}r^{1+\alpha}m_2 + (\alpha+1)(1+r)^{\alpha}r^{\alpha}m_2 + (\alpha+1)r^{\alpha}m_3 - \alpha r^{\alpha-1}m_1 - \alpha m_2(1+r)^{\alpha-1} - (\alpha+1)(1+r)^{\alpha}m_3. \tag{3.3}$$

(3.2) and (3.3) can be rewritten as

$$\begin{pmatrix} P \\ \dfrac{\partial P}{\partial r} \end{pmatrix} = A\begin{pmatrix} r^{\alpha-1} \\ -(1+r)^{\alpha-1} \end{pmatrix}, \tag{3.4}$$

where

$$A := (A_{ij}) := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.5}$$

and

$$A_{11} = ((1+r)^{\alpha+1}-1)rm_1 + (1+r)^{\alpha}r^2m_2 + r^2m_3,$$

$$A_{12} = (1+r)m_2 + (1+r)^2m_3,$$

$$A_{21} = (\alpha+1)(1+r)^{\alpha}rm_1 + ((1+r)^{\alpha+1}-1)\alpha m_1 + \alpha(1+r)^{\alpha-1}r^2m_2 + (\alpha+1)(1+r)^{\alpha}rm_2 + (\alpha+1)rm_3,$$

$$A_{22} = \alpha m_2 + (\alpha+1)(1+r)m_3.$$

Note that $((1+r)^{\alpha+1}-1)>0$ for any $r>0$ and $\alpha>0$, then $A_{ij}>0$. A simple calculation shows that

$$\begin{aligned}\det(A) = &-[(1+r)^{\alpha}r\alpha + \alpha(1+r)^{\alpha+1}r^2 + r^2(1+r)^{\alpha} + (1+r)^{\alpha}r + r^3\alpha(1+r)^{\alpha+1}]m_2^2 \\ &-[rm_3 + rm_3\alpha + r^2m_3 + r^2(1+r)^{\alpha}m_1\alpha + (1+r)^{\alpha}rm_1\alpha + m_3(1+r)^{\alpha}r + m_3r^2(1+r)^{\alpha} \\ &+ r^4m_3\alpha(1+r)^{\alpha+1} + r^2(1+r)^{\alpha}m_1 + (1+r)^{\alpha}rm_1 + \alpha((1+r)^{\alpha+1}-1)m_1 + m_3(1+r)^{\alpha}r\alpha \\ &+ 2m_3r^3\alpha(1+r)^{\alpha+1} + m_3r^2(1+r)^{\alpha}\alpha + m_3\alpha(1+r)^{\alpha+1}r^2]m_2 - [(\alpha+1)r(r+1)]m_3^2 \\ &-[(r+1)(r+\alpha r(r+1)^{\alpha+1} + \alpha((r+1)^{\alpha+1}-1))]m_1m_3 < 0\end{aligned} \tag{3.6}$$

for any $r>0, \alpha>0$ and positive m_1, m_2, and m_3. This implies that for any $\alpha>0$ and $r>0$,

$$A_{21}P(r,\alpha) - A_{11}\frac{\partial P}{\partial r}(r,\alpha) = \det(A)(1+r)^{\alpha-1} < 0. \tag{3.7}$$

Fix $\alpha \in (0,\infty)$. Let $r=r_0(\alpha)$ be the smallest $r\in(0,\infty)$ such that $P(r,\alpha)=0$. Then $\partial P(r_0(\alpha),\alpha)/\partial r>0$ from (3.7). In fact, whenever $P(r,\alpha)\geq 0$, $\partial P(r_0(\alpha),\alpha)/\partial r>0$ from (3.7). This implies that $P(r,\alpha)>0$ for all $r>r_0(\alpha)$. Finally, since $\partial P(r_0(\alpha),\alpha)/\partial r>0$ for any $\alpha\in(0,\infty)$, then the set $\Gamma=\{(\alpha,r_0(\alpha)):0<\alpha<\infty\}$ is a smooth curve in the first quadrat of the αr-plane from the implicit function theorem. □

Remark 3.2: If α is an integer (for example, Newton's law of gravitation with $\alpha=2$), then

$$P(r,\alpha) = -(m_3+m_2) - (2m_2+3m_3)r - (m_2+3m_3)r^2 + (3m_1+m_2)r^3 + (3m_1+2m_2)r^4 + (m_2+m_1)r^5$$

is a polynomial in r (as of Ref. 13, p. 276) and the sign of its coefficients only changes once. By Descartes' rule, $P(r,\alpha)=0$ has exactly one positive solution, which implies there exists exactly one CC for any given fixed order three positive masses. However, this method is not suitable for the general case $\alpha\in(0,\infty)$.

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

Then for each permutation of m_1, m_2, and m_3 on the line, there exists exactly one CC. The CCs are equivalent for the order (m_1,m_2,m_3) and its inverse (m_3,m_2,m_1) because $P(r,\alpha)=0$ for (m_1,m_2,m_3) has same positive solutions as $P(1/r,\alpha)$ has for (m_3,m_2,m_1). So we have proved the following.

Theorem 3.3: *For any $\alpha>0$ and any given positive masses $m_1>0,m_2>0,m_3>0$, there exist exactly $3!/2=3$ CCs. More precisely, there is exactly one CC for each fixed order masses on the line.*

Remark 3.4: If α is not positive, then Theorem 3.3 is not true.

Example 1: When $\alpha=-1$, the real solution r of (2.1) is arbitrary for $\lambda=M=m_1+m_2+m_3$. Therefore there are infinite number of CCs.

Example 2: When $\alpha=-5$, $P(r,-5)=-r(1+r)P_0(r,-5)$ where

$$P_0(r,-5)=4m_1-m_2-m_3+(6m_1+m_2-4m_3)r+(4m_1-m_2-6m_3)r^2+(m_1+m_2-4m_3)r^3.$$

There are at most three possible positive r such that $P(r,-5)=0$ for given positive masses m_1, m_2, and m_3. If $m_3>4m_1+m_2/4$, then there is no positive r such that $P(r,-5)=0$ because $P_0(r,-5)$ has all negative coefficients. If m_1 is large, then there is no positive r such that $P(r,-5)=0$ because $P_0(r,-5)$ has all positive coefficients.

In particular, if $m_1=5$, $m_2=1$, and $m_3=1$, then $P(r,-5)=-r(1+r)(2r+3)(r+3)(r+2)=0$ has no positive solutions. If $m_1=1$, $m_2=5$, and $m_3=1$, then $P(r,-5)=-r(1+r)(2r-1)(r-1)(r-2)=0$ has three positive solutions at 1/2, 1, and 2. If $m_1=1$, $m_2=1$, and $m_3=5$, then $P(r,-5)=r(1+r)(3r+2)(2r+1)(3r+1)=0$ has no positive solutions. So when $\alpha=-5$ and masses 1, 1, and 5, there are three CCs but it is not true that there is exactly one CC for each fixed order masses.

If $m_1=1$, $m_2=2$, and $m_3=3$, then $P(r,-5)=r(1+r)(9r^3+16r^2+4r+1)$ has no positive zeros. If $m_1=3,m_2=2,m_3=1$, then $P(r,-5)=-r(1+r)(r^3+4r^2+16r+9)$ which has no positive zeros. If $m_1=2$, $m_2=3$¸ and $m_3=1$, then $P(r,-5)=-r(1+r)(r(r-1/2)^2+43r/4+4)$, which has no positive zeros. If $m_1=1$, $m_2=3$, and $m_3=2$, then $P(r,-5)=r(1+r)(4r^3+10r^2+(r-1/2)^2+3/4)$, which has no positive zeros. If $m_1=2$, $m_2=1$, and $m_3=3$, then $P(r,-5)=r(1+r)(9r^3+11r^2-r-4)$, which has exactly one positive zero. If $m_1=3$, $m_2=1$, and $m_3=2$, then $P(r,-5)=r(1+r)(4r^3+r^2-11r-9)$, which has exactly one positive zero. However, the two corresponding CCs are equivalent. So when $\alpha=-5$ and masses 1, 2, and 3, there is only one CC.

IV. PROOF OF THEOREM 1.5 BY INDUCTION

For any given masses $m_1,m_2,\ldots,m_n$, let $x_1,x_2,\ldots,x_n$ be the positions on the line. By a translation $\hat{x}_i=x_i-c$, the center of mass can be made at the origin (we still use x_i instead of $\hat{x}_i$). The system of equations for CC (1.3) is

$$\sum_{j=1,j\neq k}^{n}\frac{m_j(x_j-x_k)}{r_{jk}^{\alpha+1}}=-\lambda x_k,\quad 1\le k\le n. \tag{4.1}$$

Note that λ must be a positive parameter. If $(x_1,x_2,\ldots,x_n)$ is a CC for positive masses $m_1,m_2,\ldots,m_n$ and positive λ, then $(ax_1,ax_2,\ldots,ax_n)$ $(a>0)$ is also a CC for the same masses and positive parameter $\lambda/a^{\alpha+1}$ and they are equivalent CCs. To study the number of central configurations, we can fix the parameter λ. We prove Theorem 1.5 by showing that the number of real solutions of (4.1) is $n!/2$ for any positive integer $n\ge 3$. We will adapt mathematical induction that is used in Ref. 7 and it consists of the following steps.

Step 1 (Assumption). To get the induction, it is assumed that for $n=\nu$ the number of real solutions of (4.1) for $x_1,x_2,\ldots,x_\nu$ is N_ν, whatever real positive λ and positive $m_1,m_2,\ldots,m_\nu$ are given. It is known from Theorem 3.3 that when $\nu=3$ we have $N_3=3=3!/2$.

Step 2 (Infinitesimal mass extension). If an infinitesimal mass $m_{\nu+1}$ is added to the system $m_1,m_2,\ldots,m_\nu$ of positive masses, then the whole number of real solutions is

$$(\nu+1)N_\nu.$$

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

Step 3 (Positive mass extension). As the infinitesimal mass $m_{\nu+1}$ increases continuously to any finite positive value the total number of real solutions remains precisely $(\nu+1)N_\nu$.
Step 4 (Conclusion by induction). By mathematical induction, it is seen that the number of real solutions of (4.1) for $\nu+\mu$ is

$$N_{\nu+\mu} = (\nu+\mu)(\nu+\mu-1)\cdots(\nu+2)(\nu+1)N_\nu.$$

When $\nu=3$, it is known that $N_3=3!/2$. Therefore $N_{3+\mu}=(3+\mu)!/2$. Let $3+\mu=n$ and we have

$$N_n = \frac{n!}{2}. \tag{4.2}$$

To complete the mathematical induction, we only need to prove the results in steps 2 and 3.

Proof of steps 2 and 3: When there are ν finite bodies $m_1,\ldots,m_\nu$ and the infinitesimal body $m_{\nu+1}$, the system of Eq. (4.1) becomes

$$\begin{aligned}
\phi_1 &\equiv \lambda x_1 + 0 + \frac{m_2(x_2-x_1)}{r_{12}^{\alpha+1}} + \cdots + \frac{m_\nu(x_\nu-x_1)}{r_{1\nu}^{\alpha+1}} + \frac{m_{\nu+1}(x_{\nu+1}-x_1)}{r_{1\nu+1}^{\alpha+1}} = 0,\\
\phi_2 &\equiv \lambda x_2 + \frac{m_1(x_1-x_2)}{r_{21}^{\alpha+1}} + 0 + \cdots + \frac{m_\nu(x_\nu-x_2)}{r_{2\nu}^{\alpha+1}} + \frac{m_{\nu+1}(x_{\nu+1}-x_2)}{r_{2\nu+1}^{\alpha+1}} = 0,\\
&\vdots \quad \vdots \quad \vdots\\
\phi_\nu &\equiv \lambda x_\nu + \frac{m_1(x_1-x_\nu)}{r_{\nu1}^{\alpha+1}} + \frac{m_2(x_2-x_\nu)}{r_{\nu2}^{\alpha+1}} + \cdots + 0 + \frac{m_{\nu+1}(x_{\nu+1}-x_\nu)}{r_{\nu\nu+1}^{\alpha+1}} = 0,\\
\phi_{\nu+1} &\equiv \lambda x_{\nu+1} + \frac{m_1(x_1-x_{\nu+1})}{r_{\nu+11}^{\alpha+1}} + \frac{m_2(x_2-x_{\nu+1})}{r_{\nu+12}^{\alpha+1}} + \cdots + \frac{m_\nu(x_\nu-x_{\nu+1})}{r_{\nu+1\nu}^{\alpha+1}} + 0 = 0.
\end{aligned} \tag{4.3}$$

The last column of these equations is zero because $m_{\nu+1}=0$, but is written for use in the proof of the result in step 3. Consequently, the first ν equations are the equations defining the solutions when $n=\nu$. By the assumption of induction in step 1, there are N_ν real solutions for the first ν equations. Let any one of these solutions be $x_1=x_1^{(0)}, x_2=x_2^{(0)},\ldots,x_\nu^{(0)}$. Then the last equation of (4.3) becomes

$$\phi_{\nu+1} \equiv \lambda x_{\nu+1} + \frac{m_1(x_1^{(0)}-x_{\nu+1})}{r_{\nu+11}^{\alpha+1}} + \frac{m_2(x_2^{(0)}-x_{\nu+1})}{r_{\nu+12}^{\alpha+1}} + \cdots + \frac{m_\nu(x_\nu^{(0)}-x_{\nu+1})}{r_{\nu+1\nu}^{\alpha+1}} = 0. \tag{4.4}$$

We are going to prove there are precisely $\nu+1$ real solutions of (4.4). Consider $\phi_{\nu+1}$ as a function of $x_{\nu+1}$. We can easily verify that

$$\begin{aligned}
&\phi_{\nu+1}(+\infty) = +\infty,\\
&\lim_{\epsilon\to 0^+} \phi_{\nu+1}(x_j^{(0)}+\epsilon) = -\infty \quad (j=1,\ldots,\nu),\\
&\lim_{\epsilon\to 0^-} \phi_{\nu+1}(x_j^{(0)}+\epsilon) = +\infty \quad (j=1,\ldots,\nu),\\
&\phi_{\nu+1}(-\infty) = -\infty.
\end{aligned} \tag{4.5}$$

Because $\phi_{\nu+1}$ is finite and continuous except at $x_{\nu+1}=x_1^{(0)},\ldots,x_{\nu+1}=x_\nu^{(0)},+\infty,-\infty$, it follows that there is an odd number of real solutions in each of the intervals $-\infty$ to $x_p^{(0)}$, where $x_p^{(0)}$ is the

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp

smallest $x_j^{(0)}$; $x_k^{(0)}$ to $x_l^{(0)}$, where $x_k^{(0)}$ and $x_l^{(0)}$ are any to $x_j^{(0)}$ which are adjacent; and $x_q^{(0)}$ to $+\infty$, where $x_q^{(0)}$ is the largest $x_j^{(0)}$. However, the derivative of $\phi_{\nu+1}$ is

$$\frac{\partial \phi_{\nu+1}}{\partial x_{\nu+1}} \equiv \lambda + \frac{(\alpha+1)m_1}{r_{\nu+11}^{\alpha+1}} + \frac{(\alpha+1)m_2}{r_{\nu+12}^{\alpha+1}} + \cdots + \frac{(\alpha+1)m_\nu}{r_{\nu+1\nu}^{\alpha+1}} > 0, \tag{4.6}$$

which is positive except at $x_{\nu+1}=x_1^{(0)},\ldots,x_{\nu+1}=x_\nu^{(0)}$, where it is infinite. Therefore $\phi_{\nu+1}$ is an increasing function in each of the intervals, and consequently vanishes once, and only once, in each of them. Since there are $\nu+1$ of these intervals, there are, for each real solution of the first ν equations in (4.3), precisely $\nu+1$ real solutions of the last equation in (4.3). So Eq. (4.3) altogether have precisely $(\nu+1)N_\nu$ real solutions. This completes the proof for step 2.

By a very similar argument as in Ref. 7, we can prove the result in step 3. □

ACKNOWLEDGMENTS

M.W. is a senior undergraduate student and is expected to graduate on December 2009. He was partially supported by RIG Grant (Code No. 2137) 2008–2009 from Virginia State University. Z.X. was partially supported by RIG Grant (Code No. 2137) 2008–2009 from Virginia State University.

[1] Albouy, A. and Moeckel, R., "The inverse problem for collinear central configuration," Celest. Mech. Dyn. Astron. **77**, 77 (2000).
[2] Corbera, M. and Llibre, J., "Central configurations of nested regular polyhedra for the spatial 2*n*-body problem," J. Geom. Phys. **58**, 1241 (2008).
[3] Diacu, F., Pérez-Chavela, E., and Santoprete, M., "Central configurations and total collisions for quasihomogeneous *n*-body problems," Nonlinear Anal. **65**, 1425 (2006).
[4] Jones, R., "Central configurations with a quasihomogeneous potential function," J. Math. Phys. **49**, 052901 (2008).
[5] Long, Y. and Sun, S., "Collinear central configurations and singular surfaces in the mass space," Arch. Ration. Mech. Anal. **173**, 151 (2004).
[6] Moeckel, R., "On central configurations," Math. Z. **205**, 499 (1990).
[7] Moulton, F. R., "The straight line solutions of the problem of N bodies," Ann. Math. II, Ser. **12**, 1 (1910).
[8] Ouyang, T. and Xie, Z., "Collinear central confiugration in four-body problem," Celest. Mech. Dyn. Astron. **93**, 147 (2005).
[9] Ouyang, T., Xie, Z., and Zhang, S., "Pyramidal central configurations and perverse solutions," Electron J. Differ. Equations **106**, 1 (2004).
[10] Saari, D., "On the role and the properties of n body central configurations," Celest. Mech. **21**, 9 (1980).
[11] Smale, S., "Mathematical problems for the next century," Math. Intell. **20**, 7 (1998).
[12] Smale, S., "Topology and mechanics. II. The planar n-body problem," Invent. Math. **11**, 45 (1970).
[13] Wintner, A., *The Analytical Foundations of Celestial Mechanics*, Princeton Math. Series Vol. 5 (Princeton University Press, Princeton, NJ, 1952), p. 215.
[14] Xie, Z., "Central configurations of collinear three-body problem and singular surfaces in the mass space," Preprint No. 10.
[15] Xie, Z., "Inverse problem of central configurations and singular curve in the collinear 4-body problem," Preprint No. 22.
[16] Xie, Z., "Super central configurations of *n*-body problem," Preprint No. 9.
[17] Zhang, S., "Two layer nested regular polygon central configurations," Phys. Lett. A **290**, 49 (2001).
[18] Zhang, S. and Zhu, C., "Central configurations consist of two layer twisted regular polygons," Sci. China, Ser. A: Math., Phys., Astron. **45**, 1428 (2002).

Author complimentary copy. Redistribution subject to AIP license or copyright, see http://jmp.aip.org/jmp/copyright.jsp