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Super central configurations of the n-body problem

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In this paper, we consider the inverse problem of central configurations of the n-body problem. For a given $q=(q_1,q_2,\ldots,q_n)\in(\mathbf{R}^d)^n$, let $S(q)$ be the admissible set of masses by $S(q)=\{m=(m_1,\ldots,m_n)|m_i\in\mathbf{R}^+,\ q$ is a central configuration for $m\}$. For a given $m\in S(q)$, let $S_m(q)$ be the permutational admissible set about $m=(m_1,m_2,\ldots,m_n)$ by $S_m(q)=\{m'|m'\in S(q),\ m'\neq m$ and m' is a permutation of $m\}$. Here, q is called a super central configuration if there exists m such that $S_m(q)$ is nonempty. For any q in the planar four-body problem, q is not a super central configuration as an immediate consequence of a theorem proved by MacMillan and Bartky ["Permanent configurations in the problem of four bodies," Trans. Am. Math. Soc. **34**, 838 (1932)]. The main discovery in this paper is the existence of super central configurations in the collinear three-body problem. We proved that for any q in the collinear three-body problem and any $m\in S(q)$, $S_m(q)$ has at most one element and the detailed classification of $S_m(q)$ is provided.
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I. INTRODUCTION

The classical n-body problem consists of the study of the dynamics of n point masses interacting according to Newtonian gravity. We consider n particles at $q_i\in\mathbf{R}^d$ (usually with $d=1$, $d=2$, or $d=3$) with masses $m_i\in\mathbf{R}^+$, $i=1,2,\ldots,n$ and

$$m_i\ddot{q}_i=-\sum_{i\neq j}\frac{m_im_j}{|q_i-q_j|^3}(q_i-q_j). \tag{1.1}$$

When we study homographic solutions of the n-body problem, the motion at any fixed time must satisfy the following nonlinear algebraic equation system:

$$\lambda(q_i-c)-\sum_{j=1,j\neq i}^{n}\frac{m_j(q_i-q_j)}{|q_i-q_j|^3}=0,\quad 1\le i\le n \tag{1.2}$$

for a constant λ, where $c=(\Sigma m_iq_i)/M$ is the center of mass and $M=m_1+m_2+\cdots+m_n$ is the total mass. By the homogeneity of $U(q)$ of degree -1, we have $\lambda=U/2I>0$, where U is the Newtonian potential function

$$U=\sum_{1\le i<j\le n}\frac{m_im_j}{|q_i-q_j|}$$

and I is the moment of inertial of the system, i.e., $I=\frac{1}{2}\Sigma_{i=1}^n m_i|q_i|^2$. Because the potential is singular when two particles have the same position, it is natural to assume that the configuration avoids the collision set, which is defined by

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$$\Delta = \cup \{q = (q_1,q_2, \dots ,q_n) \in (\mathbf{R}^d)^n | q_i = q_j \text{ for some } i \neq j\}. \tag{1.3}$$

Definition 1.1: A configuration $q=(q_1,q_2,\dots,q_n) \in (\mathbf{R}^d)^n \setminus \Delta$ is a *central configuration* (CC for short) for $m=(m_1,m_2,\dots,m_n) \in (\mathbf{R}^+)^n$ if q is a solution of the system (1.2) for some constant $\lambda \in \mathbf{R}$.

The allowed system of masses which make a configuration central is called admissible. Given a configuration $q=(q_1,q_2,\dots,q_n) \in (\mathbf{R}^d)^n \setminus \Delta$, denote by $S(q)$ the admissible set of masses by

$$S(q) = \{m = (m_1,m_2, \dots ,m_n) | m_i \in \mathbf{R}^+, \ q \text{ is a central configuration for } m\}. \tag{1.4}$$

For a given $m \in S(q)$, let $S_m(q)$ be the permutational admissible set about m, denoted by

$$S_m(q) = \{m' \in S(q) | m' \neq m \text{ and } m' \text{ is a permutation of } m\}. \tag{1.5}$$

The requirements that $m' \neq m$ and m' is a permutation of m in $S_m(q)$ are necessary to exclude some trivial cases. For example, if q is a CC for $m=(m_1,m_2,m_3,\dots,m_n)$ with $m_1=m_2$, then q is also a CC for $m'=(m_2,m_1,m_3,\dots,m_n)$ but $m' \notin S_m(q)$. $S_m(q)$ is a finite set and has at most $n!-1$ elements in $S_m(q)$.

Definition 1.2: Configuration q is called a *super central configuration* (SCC for short) if there exists positive mass m such that $S_m(q)$ is nonempty.

Two configurations q and $p \in (\mathbf{R}^3)^n \setminus \Delta$ are *equivalent*, denoted by $q \sim p$, if and only if q and p differ by an SO(3) rotation, followed by a scalar multiplication. This defines an equivalent relation among elements in CCs for m. Each planar CC gives rise to a family of orbits in which each particle travels on a conic with a focus at the center of mass. There has been steady progress in understanding CCs and their dynamics. The question on the number of CCs for a given mass vector $m=(m_1,m_2,\dots,m_n)$ is still a challenge problem for the 21st century's mathematicians (see Ref. 22). The finiteness was proved for $n=4$ by Hampton and Moeckel,[13] and it is still open for general n. Some partial results of CCs are given in Refs. 6–8, 14, 15, 20, 21, 25, and 27 for the four-body problem with some equal masses, in Refs. 2, 18, and 23 for the five-body problem, in Refs. 3, 17, and 28–30 for the regular polygon or regular polyhedra configurations, and in Refs. 4 and 24 for general homogeneous or quasihomogeneous potentials. The existence of different types of CCs can be found in Refs. 16, 18, and 31 and references therein. For the importance and additional properties of CCs and related topics, we refer to the works of Moeckel,[11] Saari,[19] and the books.[9,10]

It is natural to consider the inverse problem: given a configuration, find mass vectors, if any, for which it is a CC. Moulton[12] considered the inverse problem for collinear n-body problem. His results depend on whether n is even or odd. Alouby and Moeckel[1] also considered the set $S(q)$ for q is a collinear n-body configuration. They proved that a given configuration determines a two-parameter family of masses, making it central when the center of mass is not fixed in advance. Ouyang and Xie[14] found the region for collinear four-body configurations for which there exists a positive mass vector making it central.

The motivation to study the set $S_m(q)$ emanates from the example of the equilateral triangle configuration[5] in the planar three-body problem. If q is the equilateral triangle configuration and $m=(m_1,m_2,m_3)$, then q is also a CC for each permutation of m. Therefore, for three distinct masses, the set $S_m(q)$, which has five elements, consists of all the permutations of (m_1,m_2,m_3). The main result of this paper is the existence of SCC in the collinear three-body problem. We consider the collinear three-body configuration $q=(q_1,q_2,q_3)$. Because CC is invariant up to translation and scaling, we can choose the coordinate system so that all the three bodies are on the x-axis with positions $q_1=0$, $q_2=1$, and $q_3=1+r$, where $r>0$. This is a general form of collinear three-body configuration. Denote the following positive functions in $(\underline{r},\bar{r})$ by

$$f_1(r) = 1 + 4r + 6r^2 + 3r^3 - 3r^4 - 3r^5 - r^6,$$

$$f_2(r) = r^6 + 4r^5 + 6r^4 + 3r^3 - 3r^2 - 3r - 1,$$

$$f_3(r) = 1 + 2r + r^2 - r^3 + r^4 + 2r^5 + r^6,$$

$$g(r) = (1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1), \tag{1.6}$$

where $\underline{r}$ is the unique positive root of $f_2(r)=0$ and $\bar{r}=1/\underline{r}$. Numerically, $\underline{r}=0.787\,516\,154\,2$ and $\bar{r}=1.269\,815\,222$.

Theorem 1.3: *Let $r>0$ and $q=(0,1,1+r)$.*

(1) For any $m=(m_1,m_2,m_3)\in S(q)$, either ${}^{\#}S_m(q)=0$ or ${}^{\#}S_m(q)=1$.
(2) ${}^{\#}S_m(q)=1$ only in the following four cases:

- *(i) If $m_2<m_3<m_1$, then $\underline{r}<r<1$, $m_1=Mf_1(r)/g(r)$, $m_2=Mf_2(r)/g(r)$, $m_3=Mf_3(r)/g(r)$, and $S_m(q)=\{(m_3,m_1,m_2)\}$.*
- *(ii) If $m_1<m_3<m_2$, then $1<r<\bar{r}$, $m_1=Mf_1(r)/g(r)$, $m_2=Mf_2(r)/g(r)$, $m_3=Mf_3(r)/g(r)$, and $S_m(q)=\{(m_3,m_1,m_2)\}$.*
- *(iii) If $m_3<m_1<m_2$, then $\underline{r}<r<1$, $m_1=Mf_3(r)/g(r)$, $m_2=Mf_1(r)/g(r)$, $m_3=Mf_2(r)/g(r)$, and $S_m(q)=\{(m_2,m_3,m_1)\}$.*
- *(iv) If $m_3<m_1<m_2$, then $1<r<\bar{r}$, $m_1=Mf_3(r)/g(r)$, $m_2=Mf_1(r)/g(r)$, $m_3=Mf_2(r)/g(r)$, and $S_m(q)=\{(m_2,m_3,m_1)\}$.*

Example 1.4: Let $r=0.8$ and $M=1$. Then $m_1=0.622\,788\,063\,8$, $m_2=0.021\,612\,854\,37$, and $m_3=0.355\,599\,081\,9$. It is easy to check that $q=(0,1,1+r)$ is a CC for $m=(m_1,m_2,m_3)$ with $\lambda=0.198\,530\,155\,8$ and the center of mass $c=0.661\,691\,201\,6$ from Eq. (1.2). $q=(0,1,1+r)$ is also a CC for $m=(m_3,m_1,m_2)$ with $\lambda=0.951\,287\,694\,8$ and the same center of mass $c=0.661\,691\,201\,6$ from Eq. (1.2). So $(m_3,m_1,m_2)\in S_{(m_1,m_2,m_3)}(q)$.

It is not known so far whether there exist some q in other n-body problems such that $S_m(q)$ is nonempty except the equilateral triangle configuration. In fact, $S_m(q)$ is empty for any $m\in S(q)$ if q is a configuration in planar four-body problem. This is an immediate consequence from the following result.

Lemma 1.5: (Reference 8, p. 872) Associated with each admissible quadrilateral there is one and only one set of mass ratios, with the single exception of three equal masses at the vertices of an equilateral triangle and a fourth arbitrary mass at the center of gravity of the other three.

Remark 1.6: We are working on the existence and classifications of the SCCs in the collinear four-body problem and other n-body problems.[27,26] Some surprising phenomena occur. The golden ratio, called the mathematical beauty, has some connections with SCCs of the n-body problem.

II. $S_m(q)$ IN THE COLLINEAR THREE-BODY PROBLEM

To give the proof of Theorem 1.3, we need the following notations. For any $n\in\mathbf{N}$ (the set of integers), we denote by $P(n)$ the set of all permutations of $\{1,2,\ldots,n\}$. For any element $\tau\in P(n)$, we use $\tau=(\tau(1),\tau(2),\ldots,\tau(n))$ to denote the permutation τ. We also denote a permutation of $(m_1,m_2,\ldots,m_n)$ by $m(\tau)=(m_{\tau(1)},m_{\tau(2)},\ldots,m_{\tau(n)})$ for $\tau\in P(n)$. We define the converse permutation of τ by $con(\tau)=(\tau(n),\ldots,\tau(1))$.

Proof of Theorem 1.3: Fix $r>0$ and the general form of the collinear three-body configuration $q=(0,1,1+r)$. To get all the three-body collinear CCs, we use (1.2) and choose $m=(m_1,m_2,m_3)$ with m_i attached to q_i for $i=1,2,3$. Substitute the center of mass $c=(m_2+m_3(1+r))/M$ into (1.2), where M is the total mass which is chosen as a parameter. Then (1.2) is equivalent to

$$m_2\left(-1+\frac{\lambda}{M}\right)+m_3\left(-(1+r)^{-2}+\frac{\lambda(1+r)}{M}\right)=0, \tag{2.1}$$

$$m_1+\frac{m_2\lambda}{M}+m_3\left(-r^{-2}+\frac{\lambda(1+r)}{M}\right)=\lambda, \tag{2.2}$$

$$\frac{m_1}{(1+r)^2} + m_2\left(r^{-2} + \frac{\lambda}{M}\right) + \frac{m_3\lambda(1+r)}{M} = \lambda(1+r). \tag{2.3}$$

By the method of Gaussian elimination, we can find solutions of Eqs. (2.1)–(2.3) with parameters (r,M,λ),

$$m_1 = (1+r)^2(M-\lambda r^3)/d(r),$$

$$m_2 = r^2(-M+\lambda(1+r)^3)/d(r),$$

$$m_3 = r^2(1+r)^2(M-\lambda)/d(r). \tag{2.4}$$

where $d(r)=r^4+2r^3+r^2+2r+1$. Remarkably, the center c of mass only depends on r

$$c = r^3(3+3r+r^2)/d(r). \tag{2.5}$$

Albouy and Moeckel had the above solutions in Ref. 1. When

$$\max(1,r^3) < M/\lambda < (1+r)^3,$$

all three masses m_1, m_2, and m_3 are positive. So we can choose appropriate M and λ in an open interval such that $m \in S(q)$. Now we turn to study the set of $S_m(q)$ for a given $m=(m_1,m_2,m_3) \in S(q)$. Because $S_m(q)$ is a subset of $\{m(\tau) | \tau \in P(3)\}$ and $m \notin S_m(q)$, we only need to check whether other five permutations of mass m are also in $S(q)$. Because $q=(0,1,1+r)$ is fixed, when we say $m(\tau) \equiv (m_{\tau(1)},m_{\tau(2)},m_{\tau(3)}) \in S(q)$ with some $\tau \in P(3)$, we always mean that $m_{\tau(i)}$ is attached to q_i for all $i=1,2,3$.

Claim 1: If $r=1$, i.e., $q=(0,1,2)$, $S_m(q)$ is an empty set for any $m \in S(q)$.

In fact, if $r=1$, then $m_1=4(M-\lambda)/7$, $m_2=(8\lambda-M)/7$, and $m_3=4(M-\lambda)/7$ from Eq. (2.4). So m_1 must be equal to m_3. Therefore (m_1,m_3,m_2) is not in $S_m(q)$, otherwise $m_1=m_2$. Similarly $(m_2,m_1,m_3) \notin S_m(q)$. This proves the claim.

Because of Claim 1 and the uniqueness of CC for given order of mass m, $S_m(q)$ is an empty set if m_1, m_2, and m_3 are not mutually distinct.

Let $r \neq 1$ and m_1, m_2, and m_3 are mutually distinct. Suppose $m=(m_1,m_2,m_3) \in S(q)$. Denote the six permutations in $P(3)$ by

$$\tau_1 = (1,2,3), \quad \tau_2 = (3,1,2), \quad \tau_3 = (2,3,1),$$

$$\tau_4 = (1,3,2), \quad \tau_5 = (2,1,3), \quad \tau_6 = (3,2,1).$$

Claim 2: $m(\tau_4)$, $m(\tau_5)$, and $m(\tau_6)$ are not in $S_m(q)$.

Note that the center of mass c is fixed for a given r by (2.5). The center of mass is $(m_2+m_3(1+r))/M$ for $m(\tau_1)$ and the center of mass is $(m_3+m_2(1+r))/M$ for $m(\tau_4)$. If $m(\tau_4) \in S_m(q)$, we have $m_2+m_3(1+r)=m_3+m_2(1+r)$, which implies that $m_2=m_3$. This contradiction proves that $m(\tau_4)$ is not in $S_m(q)$. Similar arguments prove that $m(\tau_5)$ and $m(\tau_6)$ are not in $S_m(q)$.

Claim 3: $m(\tau_2)$ and $m(\tau_3)$ cannot be in $S_m(q)$ simultaneously.

If not, the center of mass m, the center of mass $m(\tau_2)$, and the center of mass $m(\tau_3)$ should be same, i.e.,

$$m_2 + m_3(1+r) = m_1 + m_2(1+r) = m_3 + m_1(1+r),$$

which implies that $m_1=m_2$. This contradiction proves the claim.

The three claims prove that $S_m(q)=0$ or $S_m(q)=1$ for any $m \in S(q)$ and any $q=(0,1,1+r)$.

Now we study the case of $m(\tau_2) \in S_m(q)$. Because $m(\tau_2)$ is a permutation of m, $M=m_1+m_2+m_3=m_{\tau_2(1)}+m_{\tau_2(2)}+m_{\tau_2(3)}$ is a constant. If $m(\tau_2)$ is in $S_m(q)$, then $m(\tau_2)$ should be given by Eq. (2.4) with different λ, say λ_2, i.e.,

$$m_{\tau_2(1)} = (1+r)^2(M-\lambda_2 r^3)/d(r),$$

$$m_{\tau_2(2)} = r^2(-M+\lambda_2(1+r)^3)/d(r),$$

$$m_{\tau_2(3)} = r^2(1+r)^2(M-\lambda_2)/d(r). \tag{2.6}$$

By setting $m_1 = m_{\tau_2(2)}$ and $m_2 = m_{\tau_2(3)}$, we solve for λ and λ_2,

$$\lambda = \frac{M(r^5+3r^4+4r^3-2r-1)}{r^2(1+3r+4r^2+3r^3+r^4)},$$

$$\lambda_2 = -\frac{M(r^5+2r^4-4r^2-3r-1)}{r^2(1+3r+4r^2+3r^3+r^4)}. \tag{2.7}$$

When we substitute λ and λ_2 into m_3 and $m_{\tau_2(1)}$, we have $m_3 = m_{\tau_2(1)}$, which implies that $m(\tau_2)$ is a permutation of m for the above λ and λ_2. Substituting λ into (2.4), we have

$$m_1 = Mf_1(r)/g(r), \quad m_2 = Mf_2(r)/g(r), \quad m_3 = Mf_3(r)/g(r),$$

where

$$f_1(r) = 1+4r+6r^2+3r^3-3r^4-3r^5-r^6,$$

$$f_2(r) = -1-3r-3r^2+3r^3+6r^4+4r^5+r^6,$$

$$f_3(r) = 1+2r+r^2-r^3+r^4+2r^5+r^6,$$

$$g(r) = (1+2r+r^2+2r^3+r^4)(r^2+r+1). \tag{2.8}$$

We need $f_i(r) > 0$ in order to have $m_i > 0$. Note that $f_3(r) > 0$ for all $r > 0$. By Descartes' rule and intermediate theorem, $f_1(r) = 0$ has the unique positive root $\overline{r}$ and $\overline{r}$ is greater than 1 because $f_1(1) = 7$ and $f_1(\infty) = -\infty$. Similarly, $f_2(r) = 0$ has the unique positive root $\underline{r}$ and $\underline{r}$ is less than 1 because $f_2(0) = -1$ and $f_2(1) = 7$. Note that $f_1(r) = -r^6 f_2(1/r)$, we have $\overline{r} = 1/\underline{r}$. Numerically, $\underline{r} = 0.787\,516\,154\,2$ and $\overline{r} = 1.269\,815\,222$. So m_1, m_2, and m_3 are positive for any $r \in (\underline{r}, \overline{r})$. By direct computation, we have

$$m_1 - m_3 = (1-r)(2r^4+7r^3+11r^2+7r+2)rM/g(r),$$

$$m_3 - m_2 = (1-r)(2r^4+7r^3+11r^2+7r+2)M/g(r).$$

So when $\underline{r} < r < 1$, we have (a) $m_2 < m_3 < m_1$, (b) $m \in S(q)$, (c) $^{\#}S_m(q) = 1$, and

$$S_m(q) = \{m(\tau_2) = (m_3, m_1, m_2)\}.$$

When $1 < r < \overline{r}$, we have (a) $m_1 < m_3 < m_2$, (b) $m \in S(q)$, (c) $^{\#}S_m(q) = 1$, and

$$S_m(q) = \{m(\tau_2) = (m_3, m_1, m_2)\}.$$

The proof for $m(\tau_3) = (m_2, m_3, m_1) \in S_m(q)$ is very similar to the proof for $m(\tau_2) \in S_m(q)$, and thus the proof is omitted. We only give the results.

If $m(\tau_3) \in S_m(q)$, then $r \neq 1$ should be in $(\underline{r}, \overline{r})$ and

$$\lambda = -\frac{M(r^5+2r^4-4r^2-3r-1)}{r^2(1+3r+4r^2+3r^3+r^4)}, \quad \lambda_3 = \frac{M(r^5+3r^4+4r^3-2r-1)}{r^2(1+3r+4r^2+3r^3+r^4)},$$

$$m_1 = Mf_3(r)/g(r), \quad m_2 = Mf_1(r)/g(r), \quad m_3 = Mf_2(r)/g(r).$$

And when $\underline{r} < r < 1$, we have (a) $m_3 < m_1 < m_2$, (b) $m \in S(q)$, (c) ${}^{\#}S_m(q) = 1$, and

$$S_m(q) = \{m(\tau_3) = (m_2, m_3, m_1)\}.$$

When $1 < r < \bar{r}$, we have (a) $m_2 < m_1 < m_3$, (b) $m \in S(q)$, (c) ${}^{\#}S_m(q) = 1$, and

$$S_m(q) = \{m(\tau_3) = (m_2, m_3, m_1)\}.$$

This completes the proof of Theorem 1.3. □

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