Original Research Article

The complex dynamics of a diffusive prey–predator model with an Allee effect in prey

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1. Introduction

Prey–predation interactions have been a central theme for ecological studies in the past few decades (Volterra, 1926; Arditi and Ginzburg, 1989; Berryman, 1992; Kuang and Beretta, 1998; Abrams and Ginzburg, 2000; Hsu et al., 2001; Kang and Wedekin, 2013). To study the interaction between predators and their prey, it is important to determine the specific form of the functional response between prey and predator that describes the amount of prey consumed per predator per unit of time since it affects the population dynamics dramatically (Akcakaya et al., 1995). In literature, The Holling-Type II functional response (i.e., prey density dependence) has been considered as a biological meaningful functional response (Li and Kuang, 2007). Some researchers have argued that a functional response depending on the ratio of prey to predator abundance is a suitable representation of some phenomena (Arditi and Ginzburg, 1989). For example, when predators have to search for food (and therefore have to share or compete for food), a more suitable general predator–prey theory such as the ratio-dependent theory (i.e., the per capita predator growth rate of the predator depends on a function of the ratio of prey to predator abundance), should be considered. The work by Akcakaya et al. (1995) and Hsu et al. (2001) supports that the ratio-dependent predation models (also called the prey–predator models with Michaelis–Menten type functional responses) are capable of producing richer and more reasonable dynamics biologically. Among all predator–prey models, the predation systems with the Michaelis–Menten type functional responses have gained a great interest over many decades (a partial list of references may be found in Kuang and Beretta (1998), Hsu et al. (2001), Xu and Chaplain (2002), Wang et al. (2007), Meng et al. (2010), Rao and Wang (2012), Rao (2014) and the references cited therein). However, many of these studies have concentrated on the existence, uniqueness and the stability analysis of predation systems. In addition to the functional response between prey and predator, there are many other factors such as Allee effects, dispersal movements of both prey and predator, contributing greatly to the population dynamics of prey and predator. In this paper, we propose a reaction–diffusion predator–prey model with Allee effects in prey and diffusion in both prey and predator to investigate how the synergy of Allee effects and diffusion affect the spatial temporal dynamics. More specifically, we focus on the rich...
spatial-temporal dynamics of a Michaelis–Menten type prey–predator model that is distinct from others by incorporating (1) an Allee effect in prey (which can be weak or strong); and (2) diffusion in both prey and predator.

In population dynamics, Allee effect refers to the positive correlation between the population density and the per capita growth rate (Stephens et al., 1999), which is introduced by Allee (1978). The Allee effect can be caused by a number of sources such as difficulties in finding mates at small densities, reproductive facilitation, predation, environmental conditioning and inbreeding depressions. Allee effect can be classified into strong and weak cases. The strong one introduces a population threshold, and the population must surpass this threshold to grow. In contrast, a population with a weak Allee effect does not have such threshold. Detailed investigations relating to the Allee effect can be found in the papers (Petrovskii et al., 2002; Kent et al., 2003; Zhou et al., 2005; Shi and Shi vaji, 2006; Morozov et al., 2006; Celik and Duman, 2009; Wang et al., 2011; Kang and Yakubu, 2011; Kang and Castillo-Chavez, 2014).

Allee effect can have a huge impact on the population dynamics (Cai et al., 2014). For example, Petrovskii et al. (2002) showed that a fully deterministic predator–prey model with the Allee effects in prey can predict a patchy invasion under certain the parameter ranges. The model has interesting dynamical features such as the coexistence state is unstable but no stable limit cycle exists. In Kent et al. (2003), Kent et al. concluded that the predator–prey system can be stabilized by an influx of prey due to a rescue effect, and be destabilized by an outflow of an Allee effect. Zhou et al. (2005) selected two classical predator–prey systems with the Allee effects in both predator and prey populations. They showed that the Allee effect may be a destabilizing force in these predator–prey systems. Celik and Duman (2009) investigated the impact of the Allee effect (on prey population) on the stability of the positive equilibrium point for a discrete-time predator–prey system. Wang et al. (2011) considered the dynamic of a reaction–diffusion Holling type II predator–prey system with strong Allee effect in the prey population. They showed that the impact of the Allee effect increases the system spatiotemporal complexity. The Allee effect was shown to bring essential changes to the population dynamics and it has drawn considerable attention in almost every aspect of ecology and conservation, however, there is no work that has been done for the spatio-temporal dynamics of Michaelis–Menten type predator–prey model with an Allee effect in prey. One of our main motivations is to study the synergistic effects of Allee effects in prey and diffusion in both species on the prey–predator population dynamics.

Understanding the spatial patterns and the related mechanisms of interacting species has been a great interest in conservation biology and ecology. Theoretically, there are a fair amount work by using two nonlinear reaction–diffusion equations in two spatial dimensions to study the invasion dynamics of predator–prey systems (Lewis and Kareiva, 1993; Morozov et al., 2006; Wang et al., 2011). To understand the role of random mobility of the individuals or organisms on the stability and persistence of interacting species, the diffusive predator–prey models have been studied by many authors (Lin et al., 1988; Murray, 1990; Lou and Ni, 1996; Pang and Wang, 2003; Cantrell and Cosner, 2003; Baermann et al., 2007; Banerjee and Banerjee, 2012) either qualitatively or numerically. In Lou and Ni (1996), Lou and Ni dealt with a strongly-coupled parabolic prey–predator system. They investigated the effects of diffusion, self-diffusion and cross-diffusion on the dynamics of the system. Baermann et al. (2007) studied a generalized predator–prey system on a spatial domain where diffusion is considered as the principal process of motion. Banerjee and Banerjee (2012) considered a modified spatiotemporal ecological system originating from the Holling-Tanner model by incorporating diffusion terms with both numerical and analytical approaches.

The diffusion has been observed as causes of the spontaneous emergences of ordered structures, called patterns, in a variety of nonequilibrium situations (Wen, 2013). Patterns generated in homogeneous environments are particularly interesting because they require an explanation based on the individual behavior of organisms, and they emerge from interactions in spatial scales that are much larger than the characteristic scale of individuals (Alonso et al., 2002). In mathematics, pattern formation refers to the process that, by changing a bifurcation parameter, the spatially homogeneous steady states lose stability to spatially inhomogeneous perturbations, and stable inhomogeneous solutions arise (Wang and Hillen, 2007). In recent years, many researchers show that the reaction–diffusion predator–prey model is an appropriate tool for investigating the fundamental mechanism of complex spatiotemporal predation dynamics (see Alonso et al., 2002; Baermann et al., 2007; Wang et al., 2007; Banerjee and Petrovskii, 2011; Rodrigues et al., 2011 and the references therein). There is a limited work on a reaction–diffusion model with the Michaelis–Menten type functional response incorporating an Allee effect in prey population. In this work, we study a Michaelis–Menten type predator–prey interaction model where random movements of both species are described by the diffusion terms and prey has Allee effects ranging from weak to strong. Our model uses the traditional framework of reaction–diffusion equations where the reaction part follows the Michaelis–Menten type interaction between prey and predator population. We assume that both prey and predator are capable to diffuse over a two dimensional landscape. We aim to answer the following questions through our analytic and numerical results:

1. What are the synergistic effects of diffusion and Allee effects on the spatiotemporal dynamics of our model?
2. Can our proposed model generate distinct spatial patterns? And what are the mechanisms generating these different patterns?

The organization of this paper is as follows. In Section 2, we derive a reaction–diffusion Michaelis–Menten type predator–prey model with an Allee effect in prey. In Section 3, we carry out the analysis of the basic dynamics of the model, and prove a prior estimate for positive steady states of the model. In Section 4, we analyze the stability of non-negative constant steady states, and provide sufficient conditions that guarantee the existence of Hopf–Turing bifurcation at positive constant steady states. For the comparison, we also provide the completed analysis for the corresponding ODE model. In Section 5, we provide sufficient conditions that can lead to the non-existence and existence of non-constant positive steady states. In Section 6, we use numerical simulations to reveal the emergence of different patterns and the influences of diffusion and Allee effects on the dynamical behavior of the model. We conclude our work in Section 7 and the detailed proofs of our theoretical work are given in the last section.

2. Model derivation

The two-dimensional Michaelis–Menten type prey–predator interaction model can be described as follows:

\[
\begin{align*}
\frac{dN}{dt} &= F(N) - A \frac{NP}{N + P} \\
\frac{dP}{dt} &= \mu A \frac{NP}{N + P} - M P,
\end{align*}
\]

where \(N(T)\) and \(P(T)\) are the population densities of prey and predator at time \(T \geq 0\), respectively; \(A\) is the predation rate, \(\mu\) is the
food utilization coefficient; and $M$ is the mortality rate of predator; and the function $F(N)$ describes the population dynamics of prey in the absence of predator.

There are many literature work on the traditional ODE Michaelis– Menten type predator–prey system (1) (Kuang and Beretta, 1998; Hsu et al., 2001; Berezovskaya et al., 2001; Xiao and Ruan, 2001; Tang and Zhang, 2005) where $F(N)$ is described by a logistic growth function. These authors have shown that the system has very rich dynamics. Kuang and Beretta (1998) presented global qualitative analysis of solutions, and showed that the ratio-dependent predator–prey models are rich in boundary dynamics. They also established that the system has no nontrivial periodic solutions provided the positive steady state is locally asymptotic stable. The work of Berezovskaya et al. (2001) and Xiao and Ruan (2001) showed that the origin is a complicated equilibrium point whose characteristics determine some important properties of the system. Hsu et al. (2001) showed that the limit cycle is always stable and unique once it exists. And the work of Berezovskaya et al. (2001) and Xiao and Ruan (2001) showed that the heteroclinic bifurcation plays an important role in understanding dynamics of the system.

The corresponding PDE versions of the Michaelis– Menten type predator–prey systems have gained a lot of attentions. For example, Pang and Wang (2003) studied qualitative properties of solutions of the system with the homogeneous Neumann boundary condition, including the dissipate, persistence and stability of non-negative constant steady states, and the existence of non-constant positive steady states of the system. Alonso et al. (2002) investigated how diffusion affects the stability of predator–prey coexistence equilibria and showed a new difference between ratio- and prey-dependent models. The recent work of Wang et al. (2007) and Banerjee and Petrovskii (2011) studied the spatiotemporal pattern formations in a ratio-dependent predator–prey system. Wang et al. (2007) presented a theoretical analysis of evolutionary processes that involves organisms distribution and their interaction of spatially distributed population with local diffusion. Banerjee and Petrovskii (2011) showed that the system can develop patterns both inside and outside of the Turing parameter domain.

The interesting question is how the dynamical properties of model (1) can change when prey experiences Allee effects. Following the work of Lewis and Kareiva (1993), we assume the growth function of prey with an Allee effect can be parameterized as follows:

$$F(N) = \frac{4\omega}{(K-N_0)^2} N(N-N_0)(K-N),$$

where $K$ is the prey carrying capacity; $\omega$ is the maximum per capita growth rate; and $N_0$ ($N_0 < K$) is the “Allee threshold” where the growth rate $F(N)$ becomes negative when $N < N_0$. The value of $N_0$ can be considered as a measure of the intensity of the Allee effect: the less the value of $N_0$ is, the less prominent is the Allee effect (Petrovskii et al., 2002). More specifically, the Allee effect is called ‘strong’ if $0 < N_0 < K$ (when the growth rate becomes negative for $N < N_0$) while it is called ‘weak’ if $-K < N_0 \leq 0$. When $N_0 \leq -K$, there is no Allee effect (Morozov et al., 2004).

We aim to investigate the dynamical complexity of a PDE version of the Michaelis– Menten type predator–prey with Allee effects in prey and diffusion in both prey and predator. By choosing appropriate scales for the variables of Eqs. (1) and (2), the number of parameters can be lessened. Considering dimensionless variables with the following scaling:

$$u = \frac{N}{K}, \quad v = \frac{P}{K}, \quad t = AT,$$

we obtain and employ the corresponding diffusive model as follows:

$$\begin{align*}
\frac{\partial u(x,t)}{\partial t} - D_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = au(u-b)(1-u) - uv \quad \frac{\partial v(x,t)}{\partial t} - D_2 \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = cuv \quad d v, \quad \frac{\partial v}{\partial t} + d v,
\end{align*}$$

with

$$a = \frac{4\omega K^2}{A(K-N_0)^2}, \quad b = \frac{N_0}{K}, \quad c = \mu, \quad d = \frac{M}{A},$$

where $u(x, y, t)$ and $v(x, y, t)$ are the densities of prey and predator, respectively, at moment $t$ and position $(x, y)$; $a$ is the relative intrinsic growth rate of prey; $b \in (-1, 1)$ denotes the Allee threshold where $b \in (-1, 0)$ indicates the weak Allee effects and $b \in (0, 1)$ indicates the strong Allee effects; $c \in (0, 1)$ represents the energy conversion rate from prey to predator; and $d$ denotes the relative death rate of predator. The non-negative constants $D_1$ and $D_2$ are the diffusion coefficients of $u$ and $v$ respectively, which imply the speed of individual movements.

In model (3), the continuous growth function considering Allee effect is expressed by the following equation:

$$f(u) = au(u-b)(1-u).$$

Allee effect expressed by (4) is strong if $b \in (0, 1)$ while it is weak if $b \in (-1, 0]$. Growth curves of the population with strong and weak Allee effect are depicted in Fig. 1. We will mainly focus on the strong Allee effects for our analytical study, i.e., $0 < b < 1$, but we will include the cases that $0 \leq b \leq 1$ for numerical studies to investigate the strength of Allee effects. Model (3) is to be analyzed with the following initial conditions:

$$u(x, y, 0) = 0, \quad v(x, y, 0) = 0, \quad (x, y) \in \Omega = [0, L_x] \times [0, L_y],$$

and zero-flux boundary conditions:

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad (x, y) \in \partial \Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$; $L_x$ and $L_y$ give the size of the model in the directions of $x$ and $y$, respectively; $n$ is the outward unit normal on $\partial \Omega$; and zero-flux boundary conditions imply that no external input is imposed from outside (Murray, 1990). In the case that $u(x, y, t) = v(x, y, t) = 0$, we obtain:

$$f(u) = au(u-b)(1-u).$$

**Fig. 1.** Growth rates of the prey population with strong $(b = 0.5)$ (the red curve) and weak $(b = 0$–the black curve; $b = -0.45$ – the blue curve) Allee effects. The green curve represents the limiting case of logistic growth. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)
define \( \frac{\partial u}{\partial n} |_{u=0, y=0} = 0 \). We would like to point out that this modification is used to treat the singularity at the origin, and that it was introduced by Kuang and Beretta (1998) as well as Xiao and Ruan (2001). In the following Sections 3–5, we will provide our theoretical results of the PDE model (3) and the corresponding ODE (10). We would like to point out that our theoretical results of Model (3) are held for the bounded domain \( \Omega \subset \mathbb{R}^2 \) for \( s \geq 2 \) but we will focus on the case of \( s = 2 \).

3. Basic dynamics

In this section, we investigate the basic dynamics of model (3) such as uniqueness of solution, positive invariance in its state space, and the boundedness.

**Theorem 3.1.** Suppose that \( a, d, D_1, D_2 > 0, 0 < b < 1, 0 < c \leq 1, \) and \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary. The following statements hold:

1. If \( u(x, y, 0) = u_0(x, y) \geq 0, v(x, y, 0) = v_0(x, y) \geq 0, \) then model (3) has a unique solution \((u(x, y, t), v(x, y, t))\) such that \( u(x, y, t) \geq 0, v(x, y, t) \geq 0 \) for \( t \in [0, \infty) \) and \((x, y) \in \Omega\).
2. If \( u_0(x, y) < b \) and \( u_0(y_0) \neq (b, 0), \) then \((u(x, y, t), v(x, y, t))\) tends to \((0, 0)\) uniformly as \( t \to \infty\).
3. If \( d > c, \) then \((u(x, y, t), v(x, y, t))\) tends to \((u_0(x, y), 0)\) uniformly as \( t \to -\infty, \) where \( u_0(x, y) \) is a non-negative solution of

\[
\frac{\partial u}{\partial t} = D_1 \Delta u + au(b - u)(1 - u) - \frac{uv}{u + v}, \quad (x, y) \in \Omega, \\
\frac{\partial v}{\partial t} = D_2 \Delta v + c\frac{uv}{u + v} - dv, \quad (x, y) \in \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad (x, y) \in \partial \Omega.
\]

(7)

4. For any solution \((u(x, y, t), v(x, y, t))\) of (3),

\[
limsup_{t \to \infty} u(x, y, t) \leq 1, \quad \limsup_{t \to \infty} v(x, y, t) \leq \frac{c}{d}.
\]

Moreover, if \( D_1 = D_2, \) then for any \((x, y) \in \Omega,\)

\[
limsup_{t \to \infty} v(x, y, t) \leq \frac{\sqrt{d}}{4d} + 1.
\]

**Remark 1.** Theorem 3.1 shows that the compact set \( \Gamma := (0, 1] \times [0, \frac{c}{d}] \) is a global attractor for all solutions of model (3) in the sense that any non-negative solution \((u(x, y, t), v(x, y, t))\) of (3) enters in \( \Gamma \) for large \( t \) and for all \((x, y) \in \Omega\). This indicates that our model is well-defined biologically. The results on the dynamical behavior of model (3) in Theorem 3.1 parts (ii) and (iii) also imply the following results on the steady state solution of (3), which satisfy:

\[
\begin{align*}
-D_1 \Delta u &= au(b - u)(1 - u) - \frac{uv}{u + v}, \quad (x, y) \in \Omega, \\
-D_2 \Delta v &= c\frac{uv}{u + v} - dv, \quad (x, y) \in \Omega, \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0, \quad (x, y) \in \partial \Omega.
\end{align*}
\]

(8)

Therefore, we can have the following Corollary.

**Corollary 3.1.** Suppose that \( a, b, D_1, D_2 > 0, \) \( b \in (0, 1], \) \( c \in (0, 1], \) and \( \Omega \) is a bounded domain with smooth boundary. Let \((u(x, y), v(x, y))\) be a non-negative solution of (8):

1. If \( u(x, y) \leq b \) for all \((x, y) \in \Omega, \) then \((u(x, y), v(x, y))\) must be \((0, 0)\) or \((b, 0).\)
2. If \( d > c, \) then \((u(x, y), v(x, y))\) must be in form of \((u_0, 0), \) which called a semi-trivial solution.

**Proof.** From Theorem 3.1 parts (ii) and (iii), we only need to prove the case when \( d \geq c. \) According to Theorem 3.1 part (iv), we have \( u(x, y) \leq 1 \) and \( \frac{\partial u}{\partial n} - d \leq 0.\) By integrating the second equation of (8), we obtain follows

\[
0 \leq D_2 \int_{\Omega} \nabla v^2 d\lambda = \int_{\Omega} \nabla \left( \frac{cv}{u + v} - dv \right) d\lambda \leq 0,
\]

which indicates that \( v \equiv 0 \) on \( \Omega. \) Therefore, statement holds.

Now we provide a priori estimates for any non-negative solutions of model (8). To obtain this aim, we recall two known results that are due to Lou and Ni (1996), and to Lin et al. (1998).

**Lemma 3.1.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \) and \( g \in C^1(\Omega \times \mathbb{R}) \).

1. If \( g \in C^2(\Omega \cap C^1(\partial \Omega)) \) satisfies

\[
\Delta w(x, y) + g(x, y, w(x, y)) \geq 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial n} \leq 0 \text{ on } \partial \Omega,
\]

and \( w(x_0, y_0) = \max_{\Omega} w(x, y), \) then \( g(x_0, y_0, w(x_0, y_0)) \geq 0.\)

2. If \( g \in C^2(\Omega \cap C^1(\partial \Omega)) \) satisfies

\[
\Delta w(x, y) + g(x, y, w(x, y)) \leq 0 \text{ in } \Omega, \quad \frac{\partial w}{\partial n} \geq 0 \text{ on } \partial \Omega,
\]

and \( w(x_0, y_0) = \min_{\Omega} w(x, y), \) then \( g(x_0, y_0, w(x_0, y_0)) \leq 0.\)

The results of Lemma 3.1 and the application of the maximum principle theorem allow us to obtain the following estimates.

**Theorem 3.2.** Suppose that \((u(x, y), v(x, y))\) is a non-negative solution of model (8), then for \((x, y) \in \Omega,\)

\[
0 \leq u(x, y) < 1, \quad \text{and} \quad 0 \leq v(x, y) < \frac{2a(1-b)^2}{4ad} + \frac{D_1}{D_2},
\]

(9)

where \( a, d, D_1, D_2 > 0 \) and \( 0 < b < 1, \) \( 0 < c \leq 1.\)

**Remark 2.** The results of Theorem 3.2 will be used for the proof of the non-existence and existence of the non-constant positive solutions of model (8) in Section 5.

4. Constant steady state solutions

First, we provide a theorem regarding the basins of attraction of the extinction equilibrium \((0, 0)\) for model (3).

**Theorem 4.1.** Suppose that \( a, d, D_1, D_2 > 0, \) \( b \in (0, 1), \) \( c \in (0, 1] \) are fixed. For a given initial value of the prey population \( u_0(x, y) \geq 0, \) there exists a constant \( v_0, \) which depends on parameters and \( u_0(x, y), \) such that when the initial predator population \( v_0(x, y) \geq v_0, \) then the corresponding solution \((u(x, y, t), v(x, y, t))\) of model (3) tends to \((0, 0)\) uniformly for \((x, y) \in \Omega \) as \( t \to \infty.\)

**Remark 3.** Theorem 4.1 implies that \((0, 0)\) is always a locally stable steady state with basin of attraction including all large \( v_0 \) for a given \( u_0 \) and parameter values. Our results indicate that for any given initial prey population, a large enough initial predator population can always lead to the extinction of both species, i.e., the convergence to the constant steady state \((0, 0).\) To further our study, we will provide complete dynamical analysis of the ODE version of model (3) in the following subsection.
4.1. Dynamics of the ODE version of model (3)

Constant steady state solutions of model (3) are the same as its ODE version, i.e., Model (10), which is shown as follows:

\[
\begin{align*}
\frac{du}{dt} &= au(u-b)(1-u) - \frac{uv}{u+v}, \\
\frac{dv}{dt} &= cvu - dv.
\end{align*}
\tag{10}
\]

Define

\[ u^*_1 = \frac{b + 1}{2} \pm \sqrt{\left(\frac{b + 1}{2}\right)^2 - \frac{4abc - c - d}{a}}, \quad v^*_1 = \frac{c - d}{d} u^*_1; \]

where \( u^*_1 < u^*_2 \) and

\[ u^* = \frac{b + 1 + \sqrt{\left(\frac{b + 1}{2}\right)^2 + \frac{8(1-c)(c-d)}{a}}}{2}. \]

Here we provide the global dynamics of model (10) as the following theorem to facilitate our study for model (3).

**Theorem 4.2.** [Dynamics of the ODE model] Model (10) always has three boundary equilibriums: \( E_0 = (0, 0), E_1 = (b, 0), E_2 = (1, 0) \). If \( c > d \), then model (10) can have two interior equilibriums \( E_i^* = (u^*_i, v^*_i), i = 1, 2 \). Provided that interior equilibriums \( E_i^* = (u^*_i, v^*_i), i = 1, 2 \) exist, we can conclude that \( b < u^*_1 < \frac{b+1}{2} < u^*_2 < 1 \), and the existence and stability conditions of these equilibriums can be listed in Table 1. Moreover, the global dynamics of model (10) can be summarized as follows:

1. If \( c > d \) and \( a > \frac{4(c-d)}{c(b-1)^2} \), then model (10) has \( E_0 = (0, 0) \) as its global attractors.
2. If \( c < d \), then model (10) has \( E_0 = (0, 0) \) and \( E_1 = (1, 0) \) as its global attractors.
3. If \( c > d \) and

\[ a > \max \left\{ \frac{4(c-d)}{c(b-1)^2}, \frac{(c-d)cd - d^4}{2bc^2} \right\}, \]

then model (10) has \( E_0 = (0, 0) \) and \( E_2 = (u^*_2, v^*_2) \) as its global attractors. 4. If \( c > d \) and

\[ a > \frac{4(c-d)}{c(b-1)^2} \quad \text{and} \quad \frac{(c-d)cd - d^4}{2bc^2} \]

then \( (u^*_2, v^*_2) \) is a source.

**Remark 4.** Kang and Castillo-Chavez (2014) studied similar ODE models of (10), where their models are in the framework of Susceptible-Infected (SI) epidemiological models, i.e., they assume that \( c = 1 \). In this case, we can conclude that if \( E_2 \) exits, then it is always locally asymptotically stable since \( u^*_2 > u^* \) always holds. Theorems 4.1 and 4.2 imply that the reaction–diffusion model (3) and its ODE version (10) have the same local stability at \( E_0 \), i.e., \( E_0 \) is always locally stable for both cases. The results of Theorem 4.2 provide us a starting point to explore the effects of spatial heterogeneity through the diffusion in both prey and predator. Moreover, in Sen et al. (2012) presented the analysis results about the emerging limit cycle. They obtained that in the model with the logistic growth, the sustainable oscillations are possible whereas they are impossible in the same system with the Allee effect regardless whether or not it is strong or weak (see discussion page 22 third paragraph in Sen et al. (2012)). Their results are also supported by our ODE model (10).

In the remaining part of this section, we assume that \( d < c \leq 1 \) and \( a > \frac{4(c-d)}{c(b-1)^2} \). Under this assumption, according to Theorem 4.2, the ODE model (10) has five non-negative constant steady state solutions: \( E_0 = (0, 0), E_1 = (b, 0), E_2 = (1, 0) \) and two positive constant solutions \( E_i^* = (u^*_i, v^*_i), i = 1, 2 \). Where \( E_0 = (0, 0) \) is a saddle point, \( E_2 = (1, 0) \) is a stable point and the positive equilibria are \( E_i^* = (u^*_i, v^*_i) \), \( i = 1, 2 \). When \( E_i^* \) is a source, \( E_1 = (b, 0) \), \( E_3 = (u^*_2, v^*_2) \), \( E_2 = (1, 0) \) or \( E_2 = (u^*_1, v^*_1) \) are saddle points, and \( E_2 = (u^*_1, v^*_1) \) can be a sink or source depending on the values of \( a \). When \( c > d \) and

\[ a = \frac{4(c-d)}{c(b-1)^2} < \frac{(c-d)cd - d^4}{2bc^2}; \]

then \( E_i^* = (u^*_i, v^*_i) \) is a source.

**Table 1**

<table>
<thead>
<tr>
<th>Equilibrium/Existence Condition for Existence/Stability Condition</th>
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<tbody>
<tr>
<td>((0, 0))</td>
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<tr>
<td>((b, 0))</td>
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<td>(E_1)</td>
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</tbody>
</table>

**Paradox of enrichment and biological control paradox:** The “paradox of enrichment” means that in one predator–one prey systems, destabilization occurs as the resource supply for the prey species increased (enrichment). This paradox was first presented by Rosenzweig (1971). Biological control is to introduce natural predators or parasites to ecosystems for pest control, which works when the pest populations are either eradicated or kept at a stable and low density equilibrium or low density cycle. Luck (1990) raised a paradox for the biological control of pests by natural enemies, when a classical predator–prey model was used to solve the biological control problem. And Luck mentioned that controlled pests do not remain in a strict equilibrium; they often appear to fluctuate because of local extinctions. An interesting problem is the **biological control paradox**. The essential paradox is that if one models a predator–pest system, via the Holling type II functional response, one cannot obtain a stable coexistence equilibrium, where the pest density is low (Deng et al., 2007). However, in reality many predators introduced for biological control purposes, are able to keep pest densities down to low levels.

Li and Kuang (2007) studied the dynamics of the Michaelis–Menten ratio-dependent predator–prey system, i.e., model (10) without Allee effect takes the form of

\[
\begin{align*}
\frac{du}{dt} &= au(1-u) - \frac{uv}{u+v}, \\
\frac{dv}{dt} &= cvu - dv.
\end{align*}
\tag{11}
\]
They obtained that the ratio-dependent predator–prey system (11) does not produce the so-called "paradox of enrichment" (Rosenzweig, 1971) and the "paradox of biological control" (Arditi and Berryman, 1991). However, in Sen et al. (2012), Sen et al. both considered the case of a strong Allee effect and the case of a weak Allee effect in the prey growth rate of a ratio-dependent predator–prey system, which are described by the following system of ODEs

\[
\begin{align*}
\frac{dx}{dt} &= u(1-x)(n-x) - \frac{cuv}{n+p}, \\
\frac{dy}{dt} &= u(1-y)(m+y) - \frac{dp}{n+p}.
\end{align*}
\]

By clearly analyzing and demonstrating, they obtained that the ratio-dependent model with the Allee effect (either strong or weak) does not exhibit the 'paradox of enrichment', but does result in 'the paradox of biological control', stating that a stable coexistence of prey and predator is impossible at prey densities much smaller than the value of the carrying capacity. This conclusion is confirmed by our ODE version model. For Model (3), the constant steady states of (3) are the same as its ODE version, i.e., model (10) (see model (12) in Sen et al. (2012) and the references therein).

In the following subsections, we will investigate the local stability of the trivial constant solutions (i.e., \( E_1 \)) of reaction–diffusion version model (3), and explore how diffusion coefficient \( D_i \), \( i = 1, 2 \) affect the local stability by comparing to its ODE model (10).

4.2. Local stability of the interior equilibrium \( E_2 \) and bifurcations

Let us denote \( \mathbf{w} = (u, v)^T \), \( \mathbf{D} = \text{diag}(D_1, D_2) \),

\[
\mathbf{F}(\mathbf{w}) = (au(1-u)(1-v), cuv) \quad \text{and} \quad \mathbf{D} = (D_1, D_2),
\]

then model (3) can be rewritten as:

\[
\begin{align*}
\frac{d\mathbf{w}}{dt} &= -\mathbf{D} \mathbf{\Delta} \mathbf{w} = \mathbf{F}(\mathbf{w}), \\
\frac{\partial \mathbf{w}}{\partial n} &= 0, \\
\mathbf{w}(x, y, 0) &= (u_0(x, y), v_0(x, y))^T, \\
(x, y) &\in \Omega, \quad t > 0.
\end{align*}
\]

Let \( \Lambda = \mu_1 < \mu_2 < \cdots \) be the eigenvalues of the operator \( -\Delta \) in \( \Omega \) with the zero-flux boundary condition, \( S(\mu_i) \) be the eigenspace corresponding to \( \mu_i \) in \( C^1(\Omega) \). Assume that conditions in Part (3) of Theorem 4.3.

\[
\begin{align*}
\text{Let } \mathbf{F}_1 &= \mathbf{F}(\mathbf{w}_1), \\
\mathbf{F}_2 &= \mathbf{F}(\mathbf{w}_2), \\
\mathbf{w}_1 &= (u_1, v_1)^T, \\
\mathbf{w}_2 &= (u_2, v_2)^T.
\end{align*}
\]

The interior equilibrium \( E_2 \) of the ODE model (10) is locally stable if conditions in Part (3) of Theorem 4.2 hold. We are interested in how the diffusion may affect the local stability of \( E_2 \). First, we have the following theorem regarding the local stability of the nontrivial constant steady state \( E_2 \) of the PDE model (3).

**Theorem 4.3.** Assume that conditions in Part (3) of Theorem 4.2 hold. For model (3), the positive constant steady state \( E_2 = (u_2, v_2) \) is locally asymptotically stable if one of the following inequalities holds:

- \( \frac{d(c-d)}{c} < \frac{a}{(b-1)^2 - \frac{4c-d}{a}} \)
\[ 0 < \frac{d(c - d)}{c} - acu_2 \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}} < \frac{D_1}{D_2} < 1 \text{ or} \]

\[ \frac{D_1}{D_2} > 1 \]

\[ \left[ \frac{d(c - d)}{c} + acu_2 \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}} - \frac{d(c - d)D_1}{D_2} \right]^2 \]

\[ < 4d(c - d)au_2^2 \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}}. \]

(16)

Remark 5. Theorem 4.3 implies that whenever the diffusion coefficient of predator is smaller than the diffusion rate of prey, or the diffusion coefficient of predator is not too large along with other conditions, the PDE model (3) preserves the stable stability of \( E_1 \). In particular, the inequality (16) will reverse its direction if the ratio of the prey’s diffusion rate \( D_1 \) to the predator’s diffusion rate \( D_2 \), i.e., \( \frac{D_1}{D_2} \) is too large or too small. In the case that the ratio \( \frac{D_1}{D_2} \) is too small, then we have

\[
\left[ \frac{d(c - d)}{c} + acu_2 \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}} - \frac{d(c - d)D_1}{D_2} \right]^2 \\
\approx \left[ \frac{d(c - d)}{c} + acu_2 \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}} \right]^2 \\
\geq 4d(c - d)au_2^2 \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}}.
\]

Hopf bifurcation and Turing bifurcation are two basic types of symmetry-breaking bifurcations for the reaction–diffusion models. They are responsible for the emergence of spatiotemporal patterns. We will investigate these two bifurcations around the constant interior equilibrium \( E_2 \) for our PDE model (3). The space-independent Hopf bifurcation breaks the temporal symmetry of a model and gives rise to oscillations that are uniform in space and periodic in time. We derive the condition of Hopf bifurcation for model (3) at the positive constant steady state \( E_2 = (u_2, v_2) \), where the coexisting homogeneous steady state loses stability through Hopf bifurcation. See the following theorem.

Theorem 4.4. For model (3), the constant steady state \( E_2 = (u_2, v_2) \) can lose stability through Hopf bifurcation at the critical value of \( a \), where

\[ a = a_H \]

\[ \begin{align*}
(c - d) & \left[ \frac{cd - c - d + 4bc}{(b - 1)^2} + \sqrt{\frac{cd - c - d + 4bc}{(b - 1)^2} + \frac{4(c - d)^2}{(b - 1)^2}} \right] \\
& = \frac{2bc^2}{2bc^2}.
\end{align*} \]

In general, \( E_2 \) can lose stability through Hopf bifurcation at mode \( k \) where \( \mu_k \) is the first one that satisfies the following equality for the given \( a \):

\[ -\mu_k(D_1 + D_2) - a \frac{(b + 1) + \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}} \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}}}{2} = 0. \]

Remark 6. The Hopf-bifurcation for ODE case occurs if and only if there exists a critical value of \( a \), i.e., \( a = a_H \), such that (i) \( \left| \text{tr}(j(E_2)) \right|_{a = a_H} = 0 \); (ii) \( \left| \det(j(E_2)) \right|_{a = a_H} > 0 \); (iii) when \( E_2 = (u_2, v_2) \) exists, the characteristic equation is \( \lambda^2 + \left( \text{det}(j(E_2)) \right)_{a = a_H} = 0 \), whose roots are purely imaginary; (iv) \( \frac{\partial}{\partial a} \left( \text{tr}(j(E_2)) \right)_{a = a_H} > 0. \) The Hopf bifurcation of the corresponding PDE model (3) occurs at the branch \( k \in \mathbb{Z} \) when \( \text{Im}(\lambda(k)) \neq 0 \), \( \text{Re}(\lambda(k)) = 0 \) where \( \lambda(k) \) satisfies the following equality:

\[ \lambda(k)^2 - \text{tr}(j_k)\lambda(k) + \det(j_k) = 0. \]

Both the condition of Hopf bifurcation of our PDE model and its ODE version are the same for \( k = 0 \). In general, the PDE model (3) can possess any periodic solution of (10) as a spatially homogeneous periodic solution, including the ones from Hopf bifurcation in Theorem 4.4. In these cases, the diffusion can play an important role for the stability of these periodic solutions. However, this is not our focus here. The related work on the Hopf bifurcation of spatially homogeneous periodic solutions can be found in the recent work by Li et al. (2011).

The Turing bifurcation breaks spatial symmetry, leading to the formation of patterns that are stationary in time and oscillatory in space. Turing instability refers to the diffusion-induced instability (Turing, 1952), which means the stability of a positive constant steady-state solution \( E_2 \) changing from stable for the spatially homogeneous version of model (25) to unstable for model (3). The instability arises due to the small heterogeneous perturbation around the homogeneous steady state. We provide the conditions of Turing instability of the solutions to model (3) in the following theorem.

Theorem 4.5. Assume that conditions in Part (3) of Theorem 4.2 hold. If the following two inequalities hold

\[ D_1 - \frac{d(c - d) - acu_2^2 \sqrt{(b - 1)^2 - \frac{4(c - d)}{ac}}}{cd(c - d)} < \frac{D_1}{D_2} < \frac{D_1}{D_2} - \frac{d(c - d)D_1}{D_2} \]

and there is a wavenumber \( k^2 \in (\mu^- - \mu^+, \mu^+) \) where

\[ \mu^+(D_1, D_2, a) = \frac{\text{tr}(jD_2)}{c} + \frac{\left( \text{tr}(jD_2 + \frac{d(c - d)(D_2 - D_1)}{D_2}) \right)^2}{2D_1D_2} - \frac{4D_1D_2\text{det}(j)}{c}, \]

\[ \mu^-(D_1, D_2, a) = -\frac{\left( \text{tr}(jD_2 + \frac{d(c - d)(D_2 - D_1)}{D_2}) \right)^2}{2D_1D_2} - \frac{4D_1D_2\text{det}(j)}{c} \]

with

\[ \text{tr}(j) = au_2^2(1 - 2au_2) + \frac{u_2^2v_2^2(1 - c)}{(u_2^2 + v_2^2)^2}, \]

\[ \text{det}(j) = \frac{ac(u_2^2)^2v_2^2[2u_2^2 - (1 + b)]}{(u_2^2 + v_2^2)^4}. \]

then the positive constant steady-state solution \( E_2 = (u_2, v_2) \) of model (3) is Turing unstable.
Remark 7. Theorem 4.5 implies that Turing instability more likely to occur if the ratio of the prey’s diffusion rate $D_1$ to the predator’s diffusion rate $D_2$, i.e., $\frac{D_1}{D_2}$ is small enough, e.g., $\frac{D_1}{D_2} < \frac{1}{2}$. This result agrees with the study of predator–prey systems by Timm and Okubo (1992), which suggest that the existence of diffusive instability in such systems may require a predator’s diffusivity being sufficiently larger when compared to the prey’s. Notice that the exact expression of Turing bifurcation curves are extremely complex, and only there exists fixed number of $k^2 \in (\mu^-, \mu^+)$ where Turing instability can occur. For demonstration, we omit the exact expression of these coefficients and give a numerical example of the Turing instability curves in Fig. 3.

We fixed values of $c = 0.3$, $d = 0.1$, $D_1 = 0.01$ for Model (3). Examples of Hopf-bifurcation and Turing instability curves of (3) for $b = 0.2$, $0.1$, $0.0$ are shown in Fig. 3(b)–(d): the red curve is the value of $a_H$ where the Hopf-bifurcation at $E_2$ occurs for $k = 0$, i.e., when $a$ is less than the value of $a_H$, there is a limit cycle emerged for $a < a_H$; the blue curve is the value of $a$ and $D_2$ where $E_2$ loses its stability of the PDE model (3), i.e., below the blue curve and the above the red curve, the $E_2$ becomes unstable for the PDE model (3) while it is locally stable for the ODE model (10). Therefore, above the Turing (blue) curve, it is stable for all pairs of $(D_2, a)$, that is, there is no diffusive instability. While below the red curve, the PDE system (3) has periodic solutions.

5. Non-constant positive steady states

The focus of this section is to provide sufficient conditions on the non-existence and existence of non-constant steady states of model (8). We first discuss the non-existence of non-constant positive steady states to model (8) by the effect of diffusions. For convenience, we write $\Theta$ instead of the collective constants $(a, b, c, d)$.

Theorem 5.1. Let $\mu_1$ be the smallest positive eigenvalue of the operator $-\Delta$ on $\Omega$ with zero-flux boundary condition, and $D_2$ be a fixed positive constant such that $D_2 > \frac{c}{\mu_1}$. Then there exists a positive constant $D_1 = D_1(\Theta, D_2)$ such that model (8) has no positive non-constant steady state provided that $D_1 > D_1(\Theta, D_2)$.

Remark 8. The result of model (8) implies the nonexistence of non-constant positive solution when the diffusion coefficients are large. In the next theorem, we provide sufficient conditions of the existence of non-constant positive solutions for model (8).

To show the existence of non-constant positive solutions for model (8), we will use the priori estimates from Theorem 3.2 and the Leray–Schauder degree theory (Nirenberg, 2001). Let $w^* = E_2$, $X$ be the space defined in (14), and define

$$X^+ = \{ w \in X | u, v > 0 \text{ on } \Omega \}.$$
Then model (8) can be written as:

$$\begin{align*}
-\Delta w &= G(w), & w \in \mathbb{X}^+, \\
\partial_n w &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

(17)

where

$$G(w) = \left( \frac{u}{D_1} \left( (a(u-b)(1-u) - v) \right) \right)^T,
$$

Define a compact operator \( F : \mathbb{X}^+ \to \mathbb{X}^+ \) by

$$\mathcal{F}(w) := (I - \Delta)^{-1}(G(w) + w),$$

where \((I - \Delta)^{-1}\) is the inverse operator of \( I - \Delta \) subject to the zero-flux boundary condition. Then \( w \) is a positive solution of model (17) if and only if \( w \) satisfies

$$\mathcal{F}(w) = 0, \quad \text{in } \mathbb{X}^+.$$

In order to apply the index theory, we consider the eigenvalue problem

$$-(I - \mathcal{F}_w(w^*))\Psi = \lambda \Psi, \quad \Psi \neq 0,$$

(18)

where \( \Psi = (\psi_1, \psi_2)^T \) and \( \mathcal{F}_w(w^*) = (I - \Delta)^{-1}(I + J) \) with

$$J = \begin{bmatrix}
1 & D_1^j u_1 \\
D_2^j u_2 & 1
\end{bmatrix},
$$

After some calculations, (18) can be rewritten as:

$$\begin{align*}
-\Delta \Psi + (\lambda + 1) \Psi &= 0, & \text{in } \Omega, \\
\partial_\nu \Psi &= 0, & \text{on } \partial \Omega, \\
\Psi &\neq 0.
\end{align*}
$$

(19)

One can see that (19) has a non-trivial solution if and only if

$$\det((\lambda + 1)I - \Delta) = 0,$$

(20)

Lemma 5.1. Suppose \( M(D_1, D_2, \mu) \neq 0 \) for all \( \mu_i \in S(\mu) \). Then

$$\text{index}(I - \mathcal{F}_w(w^*)) = (-1)^6,$$

where

$$Q = \begin{cases}
\sum_{\mu_i \in S(\mu)} m(\mu_i), & \text{if } B \cap S(\mu) \neq \emptyset, \\
0, & \text{if } B \cap S(\mu) = \emptyset.
\end{cases}$$

In particular, if \( M(D_1, D_2, \mu) > 0 \) for all \( \mu \geq 0 \), then \( \varrho = 0 \).

By applying the results of Lemma 5.1, calculating index \( (I - \mathcal{F}_w(w^*)) \), and determining the range of \( \mu \) such that \( M(D_1, D_2, \mu) < 0 \), we have the following theorem by applying the approach of Pang and Wang (2003):

**Lemma 5.2.** Assume that the conditions of Theorem 3.2 hold. Then the following statements hold:

1. Model (8) has at least one non-constant solution if the inequality (21) holds,

$$\mu \in (\mu_1, \mu_1), \mu^* \in (\mu_1, \mu_1),$$

for some \( 0 \leq i < j \), and \( \sum_{i=1}^{j-1} m(\mu_j) \) is odd.

2. If \( \frac{d(c-d)}{c^2} > \sqrt{(b-1)^2 - \frac{4(c-d)}{ac}} \),

$$\left( \frac{d(c-d)}{c} + au_2 \sqrt{(b-1)^2 - \frac{4(c-d)}{ac}} \right) > 4d,$$

(21)

Remark 9. Theorem 5.2 demonstrates that under certain parameter restrictions diffusion terms can generate non-constant positive steady states, especially when the predator diffusive coefficient \( D_2 \) is large enough.

**6. Pattern formations**

In this section, we use numerical simulations to explore the impact of diffusion and the Allee effects in prey on the spatiotemporal dynamical patterns by comparing the spatially extended model (3) with Allee effects and the traditional diffusion–reaction prey–predator model (22) without Allee effects:

$$\begin{align*}
\frac{\partial u(x, y, t)}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{au(1-u)}{u + v} - uv \frac{cuv}{u + v} v.
\end{align*}
$$

(22)

Both models are solved numerically by finite differences. All simulations run in a square spatial domain 256 x 256 where the explicit scheme is used. The numerical integration is performed by means of the forward Euler integration, with a time step of \( \Delta t = \frac{1}{100} \) and a space step of \( \Delta x = \Delta y = \frac{1}{5} \). We use the standard five-point approximation for the two-dimensional Laplacian with the zero-flux boundary conditions (Garvie, 2007). In the simulations, we observed that the distributions of prey and predator are always of the same type. Thus, we show our analysis of pattern formations of...
the distribution of prey \( u \) in this paper where we take a series of snapshots with red (blue) corresponding to the high (low) value of \( u \).

Generally, the invasion of an alien species starts with a local introduction of exotic species; thus, relevant initial conditions for Model (3) and (22) should be described by functions with a compact support where the density of one or both species at the initial moment of time is non-zero and is only inside a certain domain (Wang et al., 2011). According to the work (Garvie, 2007; Medvinsky et al., 2002), the choice of initial conditions can affect the spatiotemporal dynamics of a reaction–diffusion model in ecosystems. In this section, we choose the following forms of the initial conditions for Model (3) and (22) to investigate its effect for Model (3).

\[
\text{Model (3)}: u(x,y,0) = u_2^*, \quad v(x,y,0) = v_2^*, \quad 90 < x, y < 110. \quad (23)
\]

\[
\text{Model (22)}: u(x,y,0) = u^*, \quad v(x,y,0) = v^*, \quad 90 < x, y < 110. \quad (24)
\]

where \( (u_2^*, v_2^*) = \left( \frac{(b_1+1)}{2}, \frac{a_1}{c} \right) \) and \( (u^*, v^*) = \left( \frac{b_1}{2}, \frac{a_1}{c} \right) \).

Initially, the entire system is placed in the stationary state \( (u_2^*, v_2^*) \) for Model (3) with Allee effects and the unique interior equilibrium \( (u^*, v^*) \) for Model (22) without Allee effects. The propagation velocity of the initial random perturbations is of the order of \( 5 \times 10^{-4} \) space units per time unit. Then the system is integrated for \( 10^4 \) time steps with a series of images saved. We fix the values for the following set of parameters:

\[
c = 0.3, \quad d = 0.1, \quad D_1 = 0.01.
\]

The intrinsic growth rate of prey \( a \) and the diffusion of predator \( D_2 \) are chosen as the controlling parameter to investigate their effects on the regimes of biological invasion in Model (22). And the diffusion of predator \( D_2 \) is chosen as the controlling parameter to investigate its effect for Model (3).

In Fig. 3, we provide the bifurcation diagram and the related points where we perform simulations for Model (3) and (22). The \( D_2 \)-a parameter space in Fig. 3 is separated by the Turing instability curve (blue) and Hopf-bifurcation curve (red) when \( c = 0.3, d = 0.1, D_1 = 0.01 \). \( \text{Fig. 3(a)} \) is for Model (22) without Allee effect: Points \( P_1 \) and \( P_2 \) are corresponding to Figs. 4 and 5, respectively; and \( P_3, P_4 \) are points corresponding to \( \text{Fig. 6(a)} \) and \( \text{(b)} \), respectively. \( \text{Fig. 3(b)} \): Points \( P_1 \) and \( P_2 \) are corresponding to Figs. 7 and 8, respectively. \( \text{Fig. 3(c)} \): Points \( P_1 \) and \( P_2 \) are corresponding to \( \text{Fig. 9(a)} \) and \( \text{(b)} \), respectively. \( \text{Fig. 3(d)} \): Points \( P_1 \) and \( P_2 \) are corresponding to \( \text{Fig. 10(a)} \) and \( \text{(b)} \), respectively.

6.1. Without Allee effects

We first consider the case when there is no Allee effect in prey, i.e., Model (22). We are interested in how the synergistic effects of the intrinsic growth rate in prey \( a \) and the diffusion \( D_2 \) affect pattern formations of the PDE model (22) by varying the values of \( a \) and \( D_2 \) and fixing \( c = 0.3, d = 0.1, D_1 = 0.01 \). Initially, the entire system is placed in the stationary state \( (u^*, v^*) \) and \( (u_2^*, v_2^*) \). The propagation velocity of the initial random perturbations is of the order of \( 5 \times 10^{-4} \) space units per time unit.

1. In Fig. 4, we show the evolution of the spatial pattern of prey population \( u \) by taking the value of parameters \( D_2 = 0.6 \) and \( a = 0.8 \) at a different evolution time. From Fig. 4(a)-(b), one can see that a target wave pattern emerges after perturbation of the stationary solution \( u^* \) and \( v^* \) of model (22). Some iterations later it breaks from the corners and wave patterns occur (c.f., Fig. 4(c) and (d)). Then, it grows slightly and the wave increases with time (c.f., Fig. 4(e)). When the iteration time is large enough, we observe that the time-independent striped-like patterns settle down and staying stable (c.f., Fig. 4(f)). It finally prevails over the whole domain at \( t = 3000 \).

2. When the parameters are taken as \( (D_2, a) = (0.6, 0.86) \), Fig. 5 shows snapshots of the spatial distribution of model (22) at time \( t = 0, 200, 2450, 3050, 5000 \) and 9000. After perturbing the stationary solution \( u^* \) and \( v^* \), a spot pattern breaks towards the...
core (c.f., Fig. 5(b)). As time increasing, the time-independent regular spots pattern takes a while to settle down and dominates the domain eventually (c.f., Fig. 5(f)).

3. When increasing $D_2 = 0.6$ to $D_2 = 1.2$, Fig. 6 show two typical patterns for model (22) at $t = 3000$ with $a = 0.8$, and $a = 0.86$, respectively. In the case of $(D_2, a) = (1.2, 0.8)$, a symmetrical stripe pattern appears (c.f., Fig. 6(a)), and the dynamics of the model do not undergo any further changes. While increasing $a = 0.8$ to $a = 0.86$, i.e., $(D_2, a) = (1.2, 0.86)$, the spots-target wave mixtures pattern emerges and dominates the domain, see Fig. 6(b).

### 6.2. With Allee effects

In this subsection, we explore the spatiotemporal dynamics of model (3) with an Allee effect (weak or strong Allee effect). Our simulations show that the dynamical patterns are sensitive with respect to the values of $a$, $b$, $c$, i.e., we can observe a large variety of distinct patterns by making small changes in their values. We are interested in how diffusion affects pattern formations of Model (3) with Allee effects for fixed deterministic parameters and its extent to which diffusion is capable of changing the patterns exhibited by the model. We vary the values of parameters $a$, $D_2$, and fix the other parameters as $c = 0.3$, $d = 0.1$, $D_1 = 0.01$:

1. **Case (I)** $0 < b = 0.2$. In this case, the ODE model (10) has as strong Allee effect in prey with a locally asymptotically stable interior equilibrium.

   Choose $(D_2, a) = (0.6, 4.2)$. Fig. 7 shows that the spontaneous formation of complex pattern structure emerges: one can see that for model (3), the random initial distribution around the steady state $(u_0, v_0)$ leads to the formation of a strongly regular pattern in the domain (c.f., Fig. 7(c)).

   After the regular pattern forms (c.f., Fig. 7(d) and (e)), the dynamics of the model do not undergo any further changes (c.f., Fig. 7(f)).

   As $D_2$ is increased from 0.6 to 1.2, we found that the transition of different type patterns of the prey population in two dimensional space will emerge. In Fig. 8, we show the snapshots of contour pictures of the time evolution of prey
with $(D_2, a) = (1.2, 4.2)$. Although the dynamics of the model preceding the formation of spatial structure is somewhat regular like the previous case, it follows a different scenario.

2. Case (II) $b = 0$. Fig. 9 shows two typical patterns for model (3) at $t = 3000$ with different values of $a$ and $D_2$. In Fig. 9(a), $(D_2, a) = (0.6, 4.2)$, and Fig. 9(b), $(D_2, a) = (1.2, 2.703)$.

3. Case (III) $b = 0.2 < 0$ (weak Allee effects). In Fig. 10, we show two typical patterns for model (3) at $t = 3000$ with different values of $a$ and $D_2$. In the case of $(D_2, a) = (0.6, 2.25)$, a symmetrical block pattern appears and dominates the domain (c.f., Fig. 10(a)). While increasing $D_2 = 0.6$ to $D_2 = 1.2$, i.e., $(D_2, a) = (1.2, 2.25)$, the cyclic pattern emerges and the dynamics of the model do not undergo any further changes, see Fig. 10(b).

Summary on pattern formations: In this section, we use numerical simulations to explore how Allee effects, the diffusion rate, and the growth rate of the prey affect pattern formations. Our simulations show that the PDE model (22) without Allee effects exhibits complicated pattern formation such as striped-like patterns, spots patterns and spots-target wave mixtures.
patterns. More specifically, we have the following two observations:

1. The effects of the prey growth rate $a$: By comparing (1) Figs. 4(f)–5(f) and (2) Fig. 6(a) and (b), it seems that increasing the value of $a$ could generate more regular patterns.

2. The effects of the diffusion rate of predator $D_2$: By comparing (1) Figs. 4(f)–6(a) and (2) Figs. 5(f)–6(a), it seems that increasing the value of $D_2$ could generate mixture patterns (e.g., stripped patterns mixing with spots patterns).

The PDE model (3) with Allee effects in prey also exhibits rich and distinct pattern formations when we vary the key parameters. More specifically, we have the following observations:

1. The effects of the prey growth rate $a$: By comparing (1) Figs. 7(f)–8(f), it seems that increasing the value of $a$ could generate similar but more smooth patterns.

2. The effects of the diffusion rate of predator $D_2$: By comparing (1) Fig. 9(a) and (b) and (2) Fig. 10(a) and (b), it seems that increasing the value of $D_2$ could generate more regular and smooth patterns.

Our simulations show that our PDE model (3) with Allee effects in prey exhibits rich and distinct pattern formations when we vary the key parameters. In addition, the strength of Allee effects, i.e., the value of $b$, also plays important role in shaping varied dynamical patterns: In both positive and negative Allee threshold $b$ cases, our PDE model (3) exhibits the different spatial patterns when the values of $a$ and $D_2$ varied. But in the zero Allee threshold $b$ case, we find that under different values of $D_2$, the patterns seem to share many similarities. These numerical findings may enrich the research of the pattern formations arising from the interplay between the Allee effects and diffusion rates in Michaelis–Menten predator–prey model. And we could infer that the impact of the Allee effects and diffusion significantly increases the model complexity.

7. Conclusion

In this paper, we investigate the dynamical complexity of a reaction diffusion Michaelis–Menten type predator–prey interaction model with an Allee effect in prey with the zero-flux boundary conditions. Our study is distinct from others (Hilker et al., 2007; Sen et al., 2012; Aguirre et al., 2014; Flores and González-Olivares, 2014) as our model incorporates: (a) An Allee effect in prey; (b) The Michaelis–Menten type functional response between prey and predator; and (c) Diffusion in both prey and predator. Our work focuses on the case when the densities of prey and predator are spatially inhomogeneous in a bounded domain subject to the zero-flux boundary condition. Our results provide useful insights in the rich spatial-temporal dynamics of the proposed model through rigorous analysis and carefully designed numerical simulations.
The values of this study have many-fold in both aspects of mathematics and biology. Our analytical results partially provide answers to the two questions proposed in the introduction: What are the effects of diffusion and Allee effects for the species on the spatiotemporal dynamics of our model? How our proposed model may generate distinct spatial patterns?

1. We first obtain the basic behavior of time dependent solutions of the model. In Theorem 3.1(iii), we show that the predator of model (3) is destined to go extinct when the death rate of predator is too large, i.e. \( d > c \). We provide a priori estimates for the positive solutions that will be used for results on the non-existence and existence of non-constant positive solution of the model (see Theorem 3.2).

2. In Theorem 4.1, we show that for large predator initial values, both species go extinct. This indicates the importance of the initial condition in the dynamical outcomes of Model (3). We further provide the completed analysis of the ODE version of our model (see Theorem 4.2). Correspondingly, for all possible homogeneous steady states involved with model (3), we have obtained the local stability conditions dependent upon the parametric restrictions (c.f., Theorem 4.3). Our results show that the coexistence steady state loses stability through Hopf and Turing bifurcation (c.f., Theorems 4.4 and 4.5) when the predator diffusive coefficient is large while the prey diffusive coefficient is small.

3. We derive results on the non-existence and existence of non-constant positive solution of the model. Theorem 5.1 implies that the nonexistence of non-constant positive solution when the diffusion coefficients are large. Existence of non-constant steady state depends upon certain parametric restrictions involving intrinsic rate constants and diffusivity of both species (c.f., Theorem 5.2).

We illustrate spatial patterns of our proposed model when the parameter values are close to the onset of instability bifurcation via numerical simulations. We observe that the trajectories of our model are sensitive to disturbance of both initial conditions and parameters. The increase of predator diffusion intensity has a dramatic impact on the diffusion pattern formations of the species with or without the Allee effect. By fixing the values of \( b, c, d \) and \( D_1 \) in the model, we perform a large number of computer simulations with different values of the intrinsic prey growth rate \( a \) and the predator diffusive coefficient \( D_2 \). Our results show that the model with or without Allee effect has complicated spatiotemporal patterns; the Allee effect strength \( b \) plays import roles in shaping varied pattern formations; and different patterns appear when the diffusion coefficient \( D_2 \) varies. In addition, some of our main findings can be summarized as follows:

1. **Diffusion and Allee effect can induce Turing instability:** From Theorem 4.5, we know that the positive constant steady state \( E^*_2 = (u^*_2, v^*_2) \) of model (3) can be Turing unstable due to the Allee effects and the diffusion rates. More specifically, under the proper values of the Allee effect strength \( b \), Turing instability is more likely to occur when the intrinsic growth rate of prey is relatively small, the predator diffusive coefficient is large and the prey diffusive coefficient is small. The proper diffusion rate of prey and predator can de-stabilize the symmetric solutions so that the model with diffusion can form the stationary Turing pattern.

2. **Model (3) exhibits complex pattern formations, and distinct pattern formations than the PDE model (22) without Allee effects.** The different patterns shown in Figs. 4–6 (no Allee effects), and Figs. 7–10 (with Allee effects) illustrate the effects of Allee effects in pattern formations. We also observe that the predator diffusion rate can control spatial dynamical patterns: by varying the predation diffusion rate, the spatial pattern can be stripes (Fig. 4), spots (Fig. 5), spots-target wave mixtures pattern (Fig. 6(b)), and regular patterns showed in Figs. 7–10.

**Comparison with the literature work:** McLellan et al. (2010) reevaluated the potential for predator-mediated Allee effects in a multi-prey system using Hollings disc equation. In addition, they used empirical data on a large herbivore to examine how grouping behavior may influence the potential for predation-mediated Allee effects. Their work predicts that Allee effects caused by predation on relatively rare secondary prey may not occur because handling time of the abundant prey dominates the functional response such that secondary prey are largely "bycatch". However, a predator-mediated Allee effect can occur if secondary prey live in groups and if, as the population declines, their average group size declines (a relationship seen in several species) Wang et al. (2011) studied global bifurcation analysis of a general predator–prey ODE model with strong Allee effects in prey population. They showed the existence of a point-to-point heteroclinic orbit loop, considered the Hopf bifurcation, and proved the existence/uniqueness and the nonexistence of limit cycle for appropriate ranges of parameters. We have applied their results to our corresponding ODE model (10) through the work of (Kang and Castillo-Chavez, 2014).

Wang et al. (2011) investigated the dynamics of a reaction–diffusion predation model with strong Allee effects in prey and the Holling type II functional response between prey and predator (i.e., \( \frac{u}{u + c} \)) which takes the following form:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u(1-u) - \frac{u v}{a + u} \\
\frac{\partial v}{\partial t} &= d_2 \Delta v - dv + \frac{uv}{a + u}
\end{align*}
\]

with \( b > 0 \), i.e., the model of Wang et al. (2011) focuses on the prey with strong Allee effects. They obtained interesting mathematical analysis results on the global existence of solutions, the extinction of both population, a prior bounds of the dynamic and steady state solutions, the existence of nonconstant steady states and time-periodic orbits.

In our work, we study a reaction–diffusion predator–prey model with the Michaelis–Menten ratio-dependent functional response (i.e., \( \frac{u}{u + c} = \frac{u}{u + b} \)) whose per capita predator growth rate depends on the ratio of prey to predator abundances. In addition, our model allows prey population has Allee effects being weak or strong depending on the sign of \( b \), see our scaled model below

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 \Delta u + u (1-u) (u-b) - \frac{uv}{u + v} \\
\frac{\partial v}{\partial t} &= D_2 \Delta v - dv + \frac{cvv}{u + v}.
\end{align*}
\]

Our assumption on the Michaelis–Menten ratio-dependent functional response is following the work of Ackakaya et al. (1995) which focused on the ratio-dependent predation work. They showed that most natural systems are closer to the ratio-dependent functional response than to the prey dependence (i.e., the Holling type-II functional response), and the ratio-dependent models are capable of producing richer and more reasonable dynamics biologically.

In our paper, we study that the complex dynamics of the Michaelis–Menten type reaction–diffusion predation model with an Allee effect in prey. Our work focuses on the case when the densities of prey and predator are spatially inhomogeneous in a bounded domain subject to the zero-flux boundary condition. Our study provides useful insights in the rich spatial-temporal dynamics biologically.
dynamics of the proposed model through rigorous analysis and carefully designed numerical simulations. We would like to point out that some of our theoretical results using the similar approaches of Wang et al. (2011), however, we also give the boundary condition of Turing bifurcation; determine the Turing space; and investigate the pattern formations controlled by diffusion and Allee effects in the Michaelis–Menten type predation model through novel numerical simulations. The numerical results indicate that the Michaelis–Menten type predation model with diffusion and Allee effects exhibits rich dynamical transitions from the wave growth not only to striped-like/spots patterns, but also to regular patterns. This implies that the synergistic effects of diffusion and Allee effects can contribute to the patterns of the solutions in different ways.

8. Proofs

Proof of Theorem 3.1

Proof. (1) Define

\[ F(u, v) = au(u-b)(1-u) - \frac{uv}{u+v}, \quad G(u, v) = \frac{cuv}{u+v} - dv, \]

then \( F_2 \leq 0, \quad C_0 \geq 0 \) in \( \mathbb{R}^2 \), \( 0 \leq 0 \). Let \( (u(x, y, t), v(x, y, t)) = (0, 0) \) and \( (\Pi(x, y, t), \theta(x, y, t)) = (u'(t), v'(t)) \), where \( (u'(t), v'(t)) \) is a constant solution of the following ODE model:

\[
\begin{align*}
\frac{du}{dt} &= au(u-b)(1-u), \\
\frac{dv}{dt} &= -\frac{cuv}{u+v} - dv, \\
u(0) &= u' = sup u_0(x, y) \text{ and } v(0) = v' = sup v_0(x, y).
\end{align*}
\] \hfill (25)

Then \( (u(x, y, t), v(x, y, t)) = (0, 0) \) and \( (\Pi(x, y, t), \theta(x, y, t)) = (u'(t), v'(t)) \) are the lower-solution and upper-solution to (3), respectively, since

\[
\frac{\partial \pi}{\partial t} - \Delta \pi(x, y, t) - F(\Pi(x, y, t), \theta(x, y, t)) \geq \frac{\partial \theta}{\partial t} - \Delta \theta(x, y, t),
\]

and

\[
\frac{\partial \pi}{\partial t} - \Delta \pi(x, y, t) - G(\Pi(x, y, t), \theta(x, y, t)) \geq \frac{\partial \theta}{\partial t} - \Delta \theta(x, y, t).\]

The boundary conditions are satisfied, and \( 0 \leq u_0(x, y) \leq u' \) and \( 0 \leq v_0(x, y) \leq v' \). Using the definition of lower/upper-solution in Pao (1992) and Ye and Li (1994), it shows that model (3) has a unique globally defined solution \( (u(x, y, t), v(x, y, t)) \) which satisfies

\[
0 \leq u(x, y, t) \leq u'(t), \quad 0 \leq v(x, y, t) \leq v'(t), \quad t \geq 0.
\]

The strong maximum principle implies that \( u(x, y, t), v(x, y, t) > 0 \) when \( t > 0 \) for all \( (x, y) \in \Omega \). Moreover if \( u_0(x, y) \leq u' < b \), then apparently \( u(t) \to 0 \) and consequently \( v(t) \to 0 \) as \( t \to \infty \). This completes the proof of parts (i) and (ii).

(2) From the above proof, we obtain that \( u(x, y, t) \leq u'(t) \) for all \( t > 0 \). From model (25) satisfied by \( u'(t) \), one can see that \( u(t) \to 0 \) if \( u' < b \) and \( u(t) \to 1 \) if \( u' > b \). Thus for any \( e > 0 \), there exists \( T_0 > 0 \) such that \( u(x, y, t) \leq 1 + e \) in \( \Omega \) \( \times [t_0, \infty) \).

If \( d > 0 \), we choose \( e > 0 \) such that \( d > (1 + e) \), then for \( t > T_0, u(x, y, t) \leq 1 + e \). We use the comparison argument above again with \( u'(t) = u' \leq 1 + e \). Then the equation of \( v'(t) \) implies that \( 0 \leq v(x, y, t) \leq v'(t) \to 0 \) as \( t \to \infty \) uniformly for \( (x, y) \in \Omega \). The limit behavior of the equation of \( u(x, y, t) \) is determined by the semiflow generated by the scalar parabolic equation

\[
\begin{align*}
\frac{du}{dt} &= D_1 \Delta u + au(u-b)(1-u), \quad (x, y) \in \Omega, \quad t > 0, \\
\left. \frac{du}{dt} \right|_{\partial \Omega} &= 0, \quad (x, y) \in \partial \Omega.
\end{align*}
\] \hfill (26)

It is known that (26) is a gradient system, and every orbit of (26) converges to a steady state. From the theory of asymptotically autonomous dynamical systems, the solution \( \frac{u(x, y, t)}{v(x, y, t)} \) of (3) converges to \( (u_0, 0) \) as \( t \to \infty \). This proves part (iii).

(3) Since both \( u(x, y, t) \) and \( v(x, y, t) \) are positive, \( v(x, y, t) \) satisfies

\[
\begin{align*}
v_1 = D_2 \Delta v + \frac{cuv}{u+v} - dv \leq D_2 \Delta v - cu - dv.
\end{align*}
\]

By using \( limsup_{t \to \infty} u(x, y, t) \leq 1 \) proved above, then, for any \( \varepsilon > 0 \), there exists a large \( T > 0 \) such that

\[
\begin{align*}
v_1 \leq D_2 \Delta v + c(1 + e) - dv \quad \text{for} \quad (x, y, t) \in \Omega \times (T, \infty).
\end{align*}
\]

Therefore, the comparison theorem yields \( limsup_{t \to \infty} v(x, y, t) \leq c(1 + e)/d \) for \( (x, y) \in \Omega \). This estimate implies

\[
\begin{align*}
limsup_{t \to \infty} v(x, y, t) \leq \frac{c}{d} \quad \text{for} \quad (x, y) \in \Omega.
\end{align*}
\]

(4) If \( D_1 = D_2 \), we can add the two equations in (3) and obtain

\[
\begin{align*}
w_1 &= D_1 \Delta w + au(u-b)(1-u) + \frac{(1-c)uv}{u+v} - dv, \quad (x, y) \in \Omega, \quad t > T, \\
\frac{\partial w}{\partial n} &= 0, \quad (x, y) \in \partial \Omega, \quad t > T, \\
w(x, y, T) &= u(x, y, T) + v(x, y, T), \quad (x, y) \in \Omega.
\end{align*}
\]

When \( w(x, y, T) = u(x, y, T) + v(x, y, T) \). When \( t > T, u(x, y, t) \leq 1 + e \), then we get

\[
\begin{align*}
au(u-b)(1-u) + \frac{(1-c)uv}{u+v} - dv &= au(u-b)(1-u) + \frac{(1-c)uv}{u+v} - dv, \\
+ du - \frac{(a(1-b)^2 + d)(1 + e)}{4} \leq \frac{(a(1-b)^2 + d)(1 + e)}{4} - dv,
\end{align*}
\]

and for the equation

\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= D_1 \Delta \phi + \frac{(a(1-b)^2 + d)}{4}(1 + e) - d\phi, \quad (x, y) \in \Omega, \quad t > T, \\
\left. \frac{\partial \phi}{\partial n} \right|_{\partial \Omega} &= 0, \quad (x, y) \in \partial \Omega, \quad t > T. \quad \text{(27)}
\end{align*}
\]

the solution \( \phi(x, y, t) \to \frac{(a(1-b)^2 + d)}{4}(1 + e) \) as \( t \to \infty \), then the comparison argument shows that

\[
\begin{align*}
limit_{t \to \infty} \frac{u(x, y, t)}{v(x, y, t)} \leq \frac{(a(1-b)^2 + d)}{4}(1 + e), \quad \text{which implies the last part of (iv).} \quad \square
\end{align*}
\]

Proof of Theorem 3.2

Proof. Let \( (u(x, y), v(x, y)) \) be a non-negative solution of (8). If there exists \( (x_0, y_0) \in \Omega \) such that \( v(x_0, y_0) = 0 \), then \( v(x, y) \equiv 0 \) from the maximum principle and \( u(x, y, t) \) satisfies

\[
\begin{align*}
D_1 \Delta u + au(u-b)(1-u) &= 0, \quad (x, y) \in \Omega, \\
\left. \frac{du}{dt} \right|_{\partial \Omega} &= 0, \quad (x, y) \in \partial \Omega. \quad \text{(28)}
\end{align*}
\]

Similarly if \( u(x_0, y_0) = 0 \) for some \( (x_0, y_0) \in \Omega \), we also have \( u(x, y) \equiv 0 \) which implies \( v \equiv 0 \). Otherwise \( u(x, y) > 0 \) and \( v(x, y) > 0 \) for \( (x, y) \in \Omega \).
From the maximum principle, \( u(x, y) < 1 \) for all \( (x, y) \in \Omega \). By adding the two equations in (8) with parameter \( k \geq 0 \), if \( 0 \leq k \leq \frac{1}{2} \), we obtain

\[
-(D_1 \Delta u + kD_2 \Delta v) = au(u-b)(1-u) - \frac{(1-kc)uv}{u+v} - kdv \leq au(u-b)(1-u) - \frac{dD_1}{D_2} (D_1 u + kD_2 v) \leq \frac{(a-1-b^2)^2}{4d} - \frac{d}{D_2} (D_1 u + kD_2 v).
\]

Then,

\[
D_1 u + kD_2 v < \frac{(a-1-b)^2 D_2}{4d} + D_1.
\]

And if choosing \( k = \frac{1}{2} \), we can get the desired estimate. □

**Proof of Theorem 4.1**

**Proof.** For a fixed \( \epsilon > 0 \), there exists \( T_1 > 0 \) such that \( u(x, y, t) \leq 1 + \epsilon \) and \( v(x, y, t) \leq \frac{\epsilon}{2} \) for \( t > T_1 \) from Theorem 3.1 (iv). Therefore, \( u(x, y, T_1) \) satisfies

\[
\begin{align*}
    u_t &= D_1 \Delta u + au(u-b)(1-u) - \frac{uv}{u+v} \quad (x, y) \in \Omega, \quad t > T_1, \\
    u(x, y, T_1) &\leq 1 + \epsilon.
\end{align*}
\]

Let \( v_1(x, y, t) \) be the solution to

\[
\begin{align*}
    v_t &= D_2 \Delta v - dv, \\
    v(x, y) &\in \Omega, \quad t > 0, \\
    \frac{\partial v}{\partial n} |_{(x, y) = \partial \Omega, \quad t > 0}, \\
    v(x, y, 0) &= v_0(x, y).
\end{align*}
\]

From the comparison principle of parabolic equation, then \( v(x, y, t) \geq v_1(x, y, t) \) for any \( t > 0 \). Moreover, if \( v_0(x, y) \geq v_0^* \), then \( v(x, y, t) \geq v_0^* e^{-\frac{dc}{c_2} T_2} \) when \( t \geq \max \{ T_1, T_2 \} \) for some \( T_2 > 0 \).

Since \( a(u-b)(1-u) \leq \frac{2a(1-b)}{1+\epsilon} \), \( M_1 \) for all \( u \geq 0 \), and \( u + v \leq \frac{\epsilon}{2} \) for \( t > T_1 \), then \( u(x, y, t) \) satisfies that

\[
\begin{align*}
    u_t &\leq D_1 \Delta u + \left( M_1 - \frac{dD_1 e^{-\frac{dc}{c_2} T_2}}{1+\epsilon} \right) u, \\
    u(x, y, T_1) &\leq 1 + \epsilon,
\end{align*}
\]

for \( (x, y) \in \Omega \), \( T_1 < t < T_1 + T_2 \). Thus the comparison principle shows that for \( t \in \{ T_1, T_1 + T_2 \} \) and \( (x, y) \in \Omega \),

\[
u(x, y, t) \leq (1+\epsilon) \exp \left( \left( M_1 - \frac{dD_1 e^{-\frac{dc}{c_2} T_2}}{1+\epsilon} \right) (t-T_1) \right).
\]

If choosing \( v_0^* \geq 2e^{2\delta T_2} M_1 \left( 1 + \frac{1}{b} \right) \left( 1 + \frac{1}{b} \right)^{\frac{T_2}{T_1}} \), and

\[
T_2 \geq \frac{1}{M_1} \ln \left( 1 + \frac{1}{b} \right),
\]

then

\[
M_1 \frac{dD_1 e^{-\frac{dc}{c_2} T_2}}{1+\epsilon} \leq -M_1,
\]

and for any \( (x, y) \in \Omega \),

\[
(u(x, y, T_1 + T_2) \leq (1+\epsilon) \exp \left( \left( M_1 - \frac{dD_1 e^{-\frac{dc}{c_2} T_2}}{1+\epsilon} \right) (T_2-T_1) \right) < b.
\]

Therefore, \( (u(x, y, t), v(x, y, t)) \) tends to \((0, 0)\) for \( t \to \infty \) from Theorem 3.1 (ii). Since \( \epsilon \) is chosen arbitrarily, then \( v_0^* \) depends only on the fixed parameters and \( T_1 \) which depends on \( u_0(x, y) \). □

**Proof of Theorem 4.2**

**Proof.** First of all, we can easily calculate that model (25) has only three boundary equilibria: \( E_0 = (0, 0), E_1 = (0, 1), E_2 = (1, 0) \) if one of the following two conditions hold: (1) \( c \leq d \) or, (2) \( c > d \) and \( a < \frac{4(c-d)}{6-c} \). By evaluating the Jacobian matrix of model (25) at these three boundary equilibria, we have the following two cases:

- If \( c < d \), then both \( E_0 \) and \( E_2 \) are locally asymptotically stable while \( E_1 \) is a saddle. Since model (25) is a two dimensional ODE, then it has \( E_0 \) and \( E_2 \) as its global attractor according to Poincaré–Bendison theorem (Walter, 1998).
- If \( c > d \) and \( a < \frac{4(c-d)}{6-c} \), then \( E_0 \) is the only locally asymptotically stable equilibrium while \( E_1 \) is a source and \( E_2 \) is a saddle. Thus, according to Poincaré–Bendison theorem (Walter, 1998), model (25) has \( E_0 \) as its global attractor.

Now let us focus on the case when \( c > d \) and \( a > \frac{4(c-d)}{6-c} \). Under this condition, model (25) has \( E_0 = (0, 0), E_1 = (b, 0), E_2 = (1, 0) \) as its boundary equilibria where \( E_0 \) is always locally asymptotically stable, \( E_1 \) is a source and \( E_2 \) is a saddle. Model (25) has two interior equilibria \( E_i^* = (u_i, v_i), i = 1, 2 \) where \( u_i < u_2 \) and

\[
u_i = \frac{(b + 1) + \frac{4(c-d)}{6-c}}{2}.
\]

The expression of \( E_i^* \) indicates that \( b < u_i < b = \frac{1}{2} < u_2 < 1 \). The local stability of \( E_i^* \) can be determined by the Jacobian matrix of model (25) evaluating at \( E_i^* \) which has the following form:

\[
J_{E_i^*} = \begin{pmatrix}
    au_i^2 (1 + b - 2u_i) + \frac{u_i v_i}{(u_i + v_i)^2} & -u_i^2 \\
    cu_i^2 (u_i + v_i)^2 & -c u_i^2
\end{pmatrix}.
\]

Let \( \lambda_i, i = 1, 2 \) be the eigenvalues of (32), then we have the following two equalities:

\[
\begin{align*}
    \lambda_1 + \lambda_2 &= \frac{au_i^2 (1 + b - 2u_i) + \frac{u_i v_i}{(u_i + v_i)^2}}{(u_i + v_i)^2} - \frac{u_i^2}{(u_i + v_i)^2}, \\
    \lambda_1 \lambda_2 &= \frac{ac u_i v_i^2 (2u_i^2 - (1 + b))}{(u_i + v_i)^2}.
\end{align*}
\]

Notice that \( b < u_i < \frac{1}{2} < u_2 < 1 \), thus from (33), we can conclude

\[
\lambda_1 \lambda_2 (u_i^2, v_i^2) < 0 \quad \text{and} \quad \lambda_1 \lambda_2 (u_2^2, v_2^2) > 0.
\]

This implies that \( E_1 \) is always a saddle while \( E_2 \) could be a sink or source depending the value of \( u_i \). To further investigate the stability of \( E_2 \), we let \( v_2 = \frac{2a}{ac} u_i^2 \lambda \) in \( \lambda_1 + \lambda_2 \) which gives

\[
\begin{align*}
    \lambda_1 + \lambda_2 &= 2a \left[ -u_i^2 + \frac{(b + 1) u_i^2 + \frac{(1-c)(c-d)}{2ac}}{2} \right] \frac{(1-c)(c-d)}{2ac}, \\
\end{align*}
\]

Let \( f_s = -u_i^2 + \frac{(b + 1) u_i^2 + \frac{(1-c)(c-d)}{2ac}}{2} \) and \( u^* = b + 1 + \frac{\frac{(1-c)(c-d)}{2ac}}{4} \) then we can conclude follows:

\[
f_s > 0 \text{ when } 0 < u < u^*; \quad f_s < 0 \text{ when } u > u^*.
\]

This indicates that \( f_s > 0 \) if \( u_i < u^* \) while \( f_s < 0 \) if \( u_i > u^* \). Therefore, the interior equilibrium \( E_2 \) is locally asymptotically stable if \( u_i < u_2 \) while \( E_2 \) is a source if \( u_i > u_2 \) provided that \( E_2 \) exists.
To further explore sufficient conditions that lead to $E_2$ being a sink or a source, we let $u_{2}^{r} - u^{*}$ and decide how the value of $a$ affects its sign. We define $f$ as a function of $a$, i.e.,

$$f(a) = a^2 - \frac{(c-d)(cd-d-d+\frac{4bc}{b-1})}{bc^2}a - \frac{(c-d)^2cd}{bc^2(b-1)^2}.$$ 

Through algebraic calculations, we can conclude that

1. $f(a) < 0$ if $0 < a < \frac{(c-d)cd - \frac{4bc}{b-1}}{2bc^2}$, while $f(a) > 0$ if $a > \frac{(c-d)cd - \frac{4bc}{b-1}}{2bc^2}$. 

2. If $f(a) > 0$, then $u_{2}^{r} > u^{*}$, therefore, $E_2$ is a sink. While if $f(a) < 0$, then $u_{2}^{r} < u^{*}$, therefore, $E_2$ is a source.

Therefore, we can conclude that

- If $c > d$ and $a > \max\{\frac{4(c-d)}{c(b-1)^2}; \frac{4(c-d)^2}{c^2}\}$, then model (25) has $E_0 = (0, 0)$ and $E_1 = (u_{2}^{r}, v_{2}^{r})$ as its global attractors.

- If $c > d$ and $a < \max\{\frac{4(c-d)}{c(b-1)^2}; \frac{4(c-d)^2}{c^2}\}$, then $(u_{2}^{r}, v_{2}^{r})$ is source.

**Proof of Theorem 4.3**

**Proof.** From the proof of Theorem 4.2, we have $\det(J) > 0$. Since we assume that conditions in Part (3) of Theorem 4.2 hold, thus, we can conclude that when the inequality $a > \max\{\frac{4(c-d)}{c(b-1)^2}; \frac{4(c-d)^2}{c^2}\}$ holds, we have $\text{tr}(J) < 0$.

Based on the discussion on Page 8–9, we can conclude that $\det(J_i) > 0 > \text{tr}(J_i)$ holds for all $i \geq 1$ if the following inequality holds

$$\left[ a u_{2}^{r}(1 + b - 2u_{2}^{r}) \left( u_{2}^{r} + v_{2}^{r} \right) D_{2} - \frac{cu_{2}^{r}v_{2}^{r}}{(u_{2}^{r} + v_{2}^{r})^2} D_{1} \right]^2 < 4D_{1}D_{2} \frac{ac(u_{2}^{r})^2v_{2}^{r}[2u_{2}^{r} - (1 + b)]}{(u_{2}^{r} + v_{2}^{r})^2}. \tag{34}$$

From the Routh–Hurwitz criterion, the two roots $\lambda_{11}$ and $\lambda_{12}$ of $\phi_i(\lambda) = 0$ have negative real parts for $i \geq 1$.

Notice that $u_{2}^{r} = \frac{(b+1) + \sqrt{(b+1)^2 - \frac{4(c-d)}{a}}}{{a}}$, $v_{2}^{r} = \frac{c-d}{a}u_{2}^{r}$, then we have

$$2u_{2}^{r} - (1 + b) = \sqrt{(b+1)^2 - \frac{4(c-d)}{a}}, \quad \frac{u_{2}^{r}v_{2}^{r}}{(u_{2}^{r} + v_{2}^{r})^2} = \frac{d(c-d)}{c^2}.$$ 

Therefore, the inequality (34) can be rewritten as follows:

\[
\begin{align*}
\left( \frac{u_{2}^{r}v_{2}^{r}}{(u_{2}^{r} + v_{2}^{r})^2} (D_{2} - cD_{1}) - aD_{2}u_{2}^{r}(2u_{2}^{r} - (1 + b)) \right)^2 &< 4D_{1}D_{2} \frac{acu_{2}^{r}[2u_{2}^{r} - (1 + b)]}{(u_{2}^{r} + v_{2}^{r})^2} , \\
\left( \frac{d(c-d)}{c^2} (D_{2} - cD_{1}) - aD_{2}u_{2}^{r}(b+1)^2 - \frac{4(c-d)}{a} \right)^2 &< 4D_{1}D_{2} \frac{acu_{2}^{r}}{b+1} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} \frac{b+1}{c} , \\
\left( \frac{d(c-d)}{c^2} - au_{2}^{r} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} \right) D_{2} - d(c-d)D_{1} &< 4D_{1}D_{2}d(c-d)u_{2}^{r} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} , \\
\left( au_{2}^{r} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} - d(c-d) \right) D_{2} &< 4D_{1}D_{2}d(c-d)u_{2}^{r} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} ,
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\left( au_{2}^{r} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} \right)^2 &< 4D_{1}D_{2}d(c-d)u_{2}^{r} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} , \\
\frac{d^2(c-d)^2D_{2}^2}{c^4} + 2 d(c-d)D_{1}D_{2} &< 4D_{1}D_{2}d(c-d)u_{2}^{r} \sqrt{(b+1)^2 - \frac{4(c-d)}{a}} .
\end{align*}
\]
which gives:
\[
\left(\frac{d(c-d) - au'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right)^2 D_2^2 + \frac{d^2(c-d)^2 D_1^2}{c^2} - 2 d(c-d) D_1 D_2 \left(\frac{d(c-d) - au'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right) < 0
\]
\[
\left(\frac{d(c-d) + au'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right)^2 D_2^2 + \frac{d^2(c-d)^2 D_1^2}{c^2} - 2 d(c-d) D_1 D_2 \left(\frac{d(c-d) + au'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right) < 4 \left(\frac{d(c-d) D_2^2 au'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right).
\]

Therefore, we have
\[
\left[\left(\frac{d(c-d) + au'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right) D_2 - \frac{d(c-d) D_1}{c}\right]^2 < 4 \left(\frac{d(c-d) D_2^2 au'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right)
\]
which is equivalent to
\[
\left[\left(\frac{d(c-d) + cu'_2 \sqrt{b(1)^2 - 4(c-d)}}{c^2}\right) D_2 - \frac{d(c-d) D_1}{c}\right]^2 < 4d(c-d)au'_2 \sqrt{b(1)^2 - 4(c-d)}/C_3.
\]

This implies that large ratios of the prey diffusion coefficient \(D_1\) to the predator diffusion coefficient \(D_2\) can reserve the stability of \(E_2\).

On the other hand, we have \(\text{tr}(f) = au'_2 (1 + b - 2u'_2) + \frac{u'_2 (1-c)}{(u'_2 + v'_2)^2}, \quad \text{det}(f) = \frac{ac u'_2 v'_2 [2u'_2 - (1 + b)]}{(u'_2 + v'_2)^2} \). Then we have:
\[
\left(au'_2 (1 + b - 2u'_2) + \frac{u'_2 (1-c)}{(u'_2 + v'_2)^2}\right) D_2 - \frac{cu'_2 (1-c)}{(u'_2 + v'_2)^2} D_1 = \text{tr}(f) D_2 + \frac{d(c-d) (D_2 - D_1)}{c}.
\]

Thus, we can conclude that \(E_2\) is locally stable for the PDE model \((3)\) if \(\left(au'_2 (1 + b - 2u'_2) + \frac{u'_2 (1-c)}{(u'_2 + v'_2)^2}\right) D_2 - \frac{cu'_2 (1-c)}{(u'_2 + v'_2)^2} D_1 < 0\) which is equivalent to the inequality
\[
\text{tr}(f) D_2 + \frac{d(c-d) (D_2 - D_1)}{c} < 0
\]
since under this condition we have
\[
D_1 D_2 \mu_i - \left[\text{tr}(f) D_2 + \frac{d(c-d) (D_2 - D_1)}{c}\right] \mu_i + \text{det}(f) > 0.
\]

The condition of \(\text{tr}(f) D_2 + \frac{d(c-d) (D_2 - D_1)}{c} < 0\) can be held in the following three cases

1. Since Conditions in Part (3) of \textbf{Theorem 4.2} hold, then we have \(\text{tr}(f) < 0\) and \(\text{det}(f) > 0\) which implies that the inequality \((35)\) holds for all \(\mu_i > 0\) if \(D_2 < D_1\) holds.

2. \(\text{tr}(f) + \frac{d(c-d)}{c} < 0 \Leftrightarrow \frac{d(c-d)}{c} < au'_2 \sqrt{(b-1)^2 - 4(c-d)/C_3}\) since we have
\[
\text{tr}(f) = au'_2 (1 + b - 2u'_2) + \frac{u'_2 (1-c)}{(u'_2 + v'_2)^2} = \frac{d(c-d) (1-c)}{c^2} - au'_2 \sqrt{(b-1)^2 - 4(c-d)/C_3}.
\]

3. The inequalities hold \(D_1 < D_2 < \frac{d(c-d)}{\text{det}(f) - \text{tr}(f) D_2} = \frac{d(c-d) D_1}{\text{det}(f) - \text{tr}(f) D_2} \sqrt{(b-1)^2 - 4(c-d)/C_3}\) which are equivalent to
\[
0 < \frac{d(c-d) - ac^2 u'_2 \sqrt{(b-1)^2 - 4(c-d)/C_3}}{c d(c-d)} < D_1 D_2 < 1.
\]

To obtain the local stability of \(E_2\), we prove that there exists a constant \(\delta > 0\) such that
\[
\text{Re}(\lambda_i) \leq -\delta \quad \text{for all} \quad i \geq 1.
\]

Let \(\lambda = \mu_i \zeta_i\), then \(\psi(\lambda) = \mu_i^2 \zeta_i^2 - \text{tr}(J_i) \mu_i \zeta_i + \text{det}(J_i) := \psi(\zeta_i)\). Since \(\mu_i \to \infty\) as \(i \to \infty\), we can get that \(\lim_{i \to \infty} \frac{\psi(\zeta_i)}{\mu_i^2} = \zeta_i^2 + (D_1 + D_2) \zeta_i + D_1 D_2 := \psi(\zeta)\).

Therefore, \(\psi(\zeta) = 0\) has two negative roots \(-D_1\) and \(-D_2\). Let \(\delta = \min(D_1, D_2)\), then \(\text{Re}(\zeta_i) \leq -\delta\) for all \(i \geq 1\). Thus, \(\text{Re}(\lambda_i) \leq -\delta \quad \text{for all} \quad i \geq 1\). Let
\[
\delta = \max \{\text{Re}(\lambda_1), \text{Re}(\lambda_2)\}.
\]

Then \(\delta > 0\) and \((36)\) holds for \(\delta = \min(\delta, \frac{\delta}{2})\). The proof is completed. \(\square\)
Proof of Theorem 4.4

Proof. For the emergency of Hopf bifurcation at \( E_2 = (u_2^*, v_2^*) \), the matrix \( J' = -\mu_4 D + J \) must have an eigenvalue on the imaginary axis (Mei, 2000), i.e., \( \text{tr}(J') = 0 \). Assume that \( a(k) \) is the critical value of \( a \) that the Hopf bifurcation occurring at the branch \( k \) when the following equality holds

\[
-\mu_4(D_1 + D_2) - a(k) \frac{(b + 1) + \sqrt{(b-1)^2 - 4(c-d)/a}}{2} + \frac{d(c-d)(1-c)}{c^2} = 0
\]

When \( k = 0 \), the critical value of \( a \) at which Hopf bifurcation hypotheses are satisfied is

\[
a = a(0) := \frac{(c-d)}{2bc^2} \left[ \frac{4bc(d - c^2)}{b(b-1)^2} + \frac{bd^2 - b^2}{b(b-1)^2} \right].
\]

Near \( a(0) \), the complex conjugate pair \( \alpha(a) \pm i\beta(a) \) is given by:

\[
\alpha(a) = \frac{1}{2} \text{tr}(J), \quad \beta^2(a) = \text{det}(J) - \alpha^2(a).
\]

Since \( \alpha(a(0)) = 0 \) and \( \text{det}(J) > 0 \), we get \( \beta^2(a) \neq 0 \). Moreover, it is easy to check \( \frac{d}{da} \alpha(a) \neq 0 \), and therefore the statement holds. \( \square \)

Proof of Theorem 4.5

Proof. In the presence of diffusion, introducing small perturbations \( U = u - u_2^*, \ V = v - v_2^* \), where \( |U|, \ |V| \ll 1 \). To study the effect of diffusion on model (3), we have considered the linearized form of system as follows:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= J_{11} U + J_{12} V + D_1 \nabla^2 U, \\
\frac{\partial V}{\partial t} &= J_{21} U + J_{22} V + D_1 \nabla^2 V.
\end{align*}
\]

(37)

Any solution of model (37) can be expanded into a Fourier series so that

\[
\begin{align*}
U(r, t) &= \sum_{m,n=0} u_{mn}(r, t) \cos kr, \quad V(r, t) &= \sum_{m,n=0} v_{mn}(r, t) \cos kr.
\end{align*}
\]

where \( k = (k_m, k_n) \) and \( k_m = m\pi/Lx \), \( k_n = n\pi/Ly \) are the corresponding wavenumbers.

Having substituted \( u_{mn} \) and \( v_{mn} \) into (37), we obtain

\[
\begin{align*}
\frac{\partial u_{mn}}{\partial t} &= \left( J_{11} - D_1 k_m^2 \right) u_{mn} + J_{12} v_{mn}, \\
\frac{\partial v_{mn}}{\partial t} &= J_{21} u_{mn} + \left( J_{22} - D_1 k_n^2 \right) v_{mn}.
\end{align*}
\]

(38)

A general solution of (38) has the form \( C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) \), where the constants \( C_1 \) and \( C_2 \) are determined by the initial conditions (5) and the exponents \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of the following matrix:

\[
A|_{E_2} = \begin{pmatrix} J_{11} - D_1 k_n^2 & J_{12} \\ J_{21} & J_{22} - D_2 k_n^2 \end{pmatrix}.
\]

(39)

Correspondingly, \( \lambda_1 \) and \( \lambda_2 \) arise as the solution of the following equation \( \lambda^2 - \text{tr}(A) \lambda + \text{det}(A) = 0 \), or solving for \( \lambda \)

\[
\lambda_k(k) = \frac{1}{2} \left( \text{tr}_k \pm \sqrt{\text{tr}_k^2 - 4\text{det}_k} \right),
\]

(40)

where

\[
\begin{align*}
\text{tr}_k &= J_{11} + J_{22} - k_n^2(D_1 + D_2) = au_2^*(1 + b - 2u_2^*) + \frac{u_2^* v_2^*(1-c)}{(u_2^* + v_2^*)^2} - k_n^2(D_1 + D_2) \equiv \text{tr}(J) - k_n^2(D_1 + D_2), \\
\text{det}_k &= \left( J_{11} - D_1 k_n^2 \right) J_{22} - J_{12} J_{21} = D_1 D_2 k_n^2 \left[ \left( au_2^*(1 + b - 2u_2^*) + \frac{u_2^* v_2^*(1-c)}{(u_2^* + v_2^*)^2} \right) D_2 - \frac{cu_2^* v_2^*}{(u_2^* + v_2^*)^2} D_1 \right] k_n^2 + \frac{ac(u_2^*)^2 v_2^*}{(u_2^* + v_2^*)^4} [2u_2^* - (1 + b)] \\
&\equiv D_1 D_2 k_n^2 \left[ \left( au_2^*(1 + b - 2u_2^*) + \frac{u_2^* v_2^*(1-c)}{(u_2^* + v_2^*)^2} \right) D_2 - \frac{cu_2^* v_2^*}{(u_2^* + v_2^*)^2} D_1 \right] k_n^2 + \text{det}(J).
\end{align*}
\]

If the homogeneous steady state of model (3) is unstable, then one of the eigenvalues of matrix \( A \) has positive real part, which depends on the signs of trace \( \text{tr}_k \) and determinant \( \text{det}_k \) of \( A \). From Theorem 4.3, \( \text{tr}(J) < 0 \) if \( a > \frac{\sqrt{\left( \frac{4(\text{det}(J) - \text{det}(J)^2)}{\text{det}(J)^2} \right) + 4\text{det}(J)}}{2\text{det}(J)} \). Then
\[ (au_2'(1 + b - 2u_2') + \frac{u_2'v_2'}{(u_2' + v_2')^2})D_2 - \frac{cu_2'v_2'}{(u_2' + v_2')^2}D_1 > 0, \]
which is equivalent to the following inequality
\[ \frac{D_1}{D_2} < \frac{d(c-d)-acu_2'v_2'\sqrt{(b-1)^2 - \frac{4c-d}{ac}} \ }{cd(c-d)}. \]

Otherwise the equilibrium \( E_2 \) is locally stable according to Theorem 4.3 since \( tr_k < 0 \) for \( k^2 > 0 \) and \( det(f) > 0 \).

The Turing instability, mathematically speaking, as \( D_1 \ll D_2 \), occurs when \( \text{Im}(\lambda(k)) = 0, \text{Re}(\lambda(k)) = 0 \) at \( k = k_T \neq 0 \). We need \( det_k < 0 \) for some \( k^2 > 0 \) and \( det_k \) achieves its minimum below zero,
\[ \min_{k^2} det_k = \frac{4D_1D_2det(f)}{d(c-d)\left(\left(au_2'(1 + b - 2u_2') + \frac{u_2'v_2'}{(u_2' + v_2')^2}\right)D_2 - \frac{cu_2'v_2'}{(u_2' + v_2')^2}D_1\right)^2} < 0, \]
which is equivalent to the inequality of (21) holds, i.e.,
\[ \left[ \frac{d(c-d)}{c} + acu_2'v_2'(b-1)^2 - \frac{4c-d}{ac} \right] > 4d(c-d)au_2'v_2'(b-1)^2 - \frac{4c-d}{ac} \]
at the wavenumber \( k_T \) satisfies
\[ k_T^2 = \frac{\overline{det(f)}}{D_1D_2} = \frac{ac(u_2)^2v_2'(1+b) - (u_2 + v_2')^2D_1D_2}{(u_2' + v_2')^2} > 0. \]

At the Turing bifurcation threshold, the spatial symmetry of the system is broken and the patterns are stationary in time and oscillatory in space with the corresponding wavelength \( \lambda_T = 2\pi/k_T \). If (41) holds and \( \min_{k^2} det_k < 0 \) with certain \( k^2 \) in the interval of \((\mu^-, \mu^+)\) where
\[ \mu^+(D_1, D_2, a) = \frac{tr(f)D_2 + \frac{d(c-d)(D_2-D_1)}{c}}{2D_1D_2} + \sqrt{\left(\frac{tr(f)D_2 + \frac{d(c-d)(D_2-D_1)}{c}}{2D_1D_2}\right)^2 - 4D_1D_2det(f)}, \]
\[ \mu^-(D_1, D_2, a) = \frac{tr(f)D_2 + \frac{d(c-d)(D_2-D_1)}{c}}{2D_1D_2} - \sqrt{\left(\frac{tr(f)D_2 + \frac{d(c-d)(D_2-D_1)}{c}}{2D_1D_2}\right)^2 - 4D_1D_2det(f)}, \]
with
\[ tr(f) = au_2'(1 + b - 2u_2') + \frac{u_2'v_2'(1-c)}{(u_2' + v_2')^2}, \quad det(f) = \frac{ac(u_2)^2v_2'(2u_2' - (1+b))}{(u_2' + v_2')^2}. \]

Then \( E_2 \) is an unstable equilibrium with respect to model (3). Therefore, the statement holds. \( \square \)

**Proof of Theorem 5.1**

**Proof.** Let \((u, v)\) be any positive solution of model (8) and denote \( \overline{F} = |\Omega|^{-1} \int_{\Omega} f_d d\Lambda \). Multiplying the first and second equations of (8) by \((u-u)\) and \((v-v)\) integrating over \( \Omega \), respectively, and since \( u \) and \( v \) have a uniform upper bound, we have
\[
\int \int \left( D_1\frac{\partial u}{\partial t}^2 + D_2\frac{\partial v}{\partial t}^2 \right) d\Lambda = \int \int \left( \frac{\partial}{\partial t}\left( au(1-u)(u-b) - \frac{uv}{u+v} - a(1-u)\frac{u(1-u)}{u+v} - \frac{uv}{u+v} \right) \right) d\Lambda \\
+ \int \int \left( \frac{\partial}{\partial t}\left( cu(1-u)(u-b) - \frac{uv}{u+v} - a(1-u)\frac{u(1-u)}{u+v} - \frac{uv}{u+v} \right) \right) d\Lambda \\
+ \int \int \left( \frac{\partial}{\partial t}\left( cv(1-u)(u-b) - \frac{uv}{u+v} - a(1-u)\frac{u(1-u)}{u+v} - \frac{uv}{u+v} \right) \right) d\Lambda \\
+ \int \int \left( \frac{\partial}{\partial t}\left( cu(1-u)(u-b) - \frac{uv}{u+v} - a(1-u)\frac{u(1-u)}{u+v} - \frac{uv}{u+v} \right) \right) d\Lambda \\
+ \int \int \left( \frac{\partial}{\partial t}\left( cv(1-u)(u-b) - \frac{uv}{u+v} - a(1-u)\frac{u(1-u)}{u+v} - \frac{uv}{u+v} \right) \right) d\Lambda \\
\leq \int \int \left( \frac{\partial}{\partial t}\left( \frac{ab^2 + 2b + 5}{4} + cC_1 \right) \right) d\Lambda \]

for some positive constants \( C_1, C \) and an arbitrary positive constant \( \varepsilon \). Synthetically, we obtain
\[
\int \int \left( D_1u_2'(u-u)^2 + D_2u_2'(v-v)^2 \right) d\Lambda \leq \int \int \left( \frac{ab^2 + 2b + 5}{4} + c\frac{C_1}{2b^C} \right) + \left( \frac{ab^2 + 2b + 5}{4} + c\frac{C_1}{2b^C} \right) d\Lambda 
\]
by using Poincare-inequality (Wu et al., 2003). Since $D_2 \mu_1 > c$ from the assumption, we can find a sufficiently small $\varepsilon_0$ such that

$$D_2 \mu_1 \geq \frac{c}{\mu_1} (c + \frac{6c^2}{20}).$$

Finally, by taking $D_1 := \frac{1}{\mu_1} \left( \frac{\mu_1 b + \mu_2 v_1}{2} + \frac{v_2 c}{20\mu_1} \right)$, one can conclude that $u = \pi$ and $v = \nu$ which complete the proof. □

**Proof of Theorem 5.2**

**Proof.** Since the condition (21) holds, that is, $(D_1 J_{22} + D_2 J_{21})^2 > 4D_1 D_2 (J_1 J_{22} - J_1 J_{21})$, it follows that $\mu^+$ exists. On the contrary, suppose that model (8) has no non-constant positive solution. By Theorem 5.1, we can fixed $\overline{D_1} > D_1$ and $\overline{D_2} > D_2$ such that

(i) model (8) with diffusion coefficients $\overline{D_1}$ and $\overline{D_2}$ has no non-constant solutions;

(ii) $M(\overline{D_1}, \overline{D_2}, \mu) > 0$ for all $\mu \geq 0$.

From Theorem 3.2, there exists a positive constant $C^*$ such that for $\overline{D_1} \geq D_1$, $\overline{D_2} \geq D_2$, any solution $(u, v)$ of model (8) with diffusion coefficients $\overline{D_1}$ and $\overline{D_2}$ satisfies $0 < u \leq C^*$, $(x, y) \in \Omega$.

Set

$$P = \{(u, v) \in C(\overline{D_1}) \times C(\overline{D_2}) : 0 < u, v < C^* \text{ in } \overline{D_1} \},$$

and define

$$\Phi : P \times [0, 1] \rightarrow C(\overline{D_1}) \times C(\overline{D_2})$$

by

$$\Phi(w, \sigma) = (I - \Delta)^{-1}(\{G(w, \sigma) + w\},$$

where

$$G(w, \sigma) = \left(\begin{array}{c}
(\sigma D_1 + (1 - \sigma) \overline{D_1})^{-1} \left(au(u - b)(1 - u) - \frac{uv}{u + v}\right) \\
(\sigma D_2 + (1 - \sigma) \overline{D_2})^{-1} \left(\frac{cu}{u + v} - dv\right)
\end{array}\right).$$

We can see that finding the positive solution of model (17) becomes equivalent to finding the fixed point of $\Phi(w, 1)$ in $P$. By virtue of the definition of $P$, we obtain that $\Phi(w, \overline{D_1}, \overline{D_2}, \mu) = 0$ has no fixed point in $\partial P$ for $0 \leq \overline{D_1} \leq 1$.

Since $\Phi(w, t)$ is compact, the Leray–Schauder topological degree $\deg(I - \Phi(w, \overline{D_1}, \overline{D_2}, \mu), P, 0)$ is well defined. From the invariance of Leray-Schauder degree at the homotopy (Wang, 2003), we deduce

$$\deg(I - \Phi(w, 1), P, 0) = \deg(I - \Phi(w, 0), P, 0).$$

In view of $\mu^- \in (\mu_i, \mu_{i+1})$ and $\mu^+ \in (\mu_i, \mu_{i+1})$, we get $\partial(D_1, D_2) \cap S(\mu) = \{\mu_{i+1}, \mu_{i+2}, \ldots, \mu_j\}$ and $I - \Phi(w, 1) = I - \mathcal{F}$. Therefore, if model (8) has no other solutions except the constant $w^*$, then Lemma 5.1 shows that

$$\deg(I - \Phi(w, 1), P, 0) = \text{index}(I - \mathcal{F}, w^*) = (-1)^{\sum_{i=1}^{m(\mu_i)} - 1} = -1.$$

On the contrary, by the choice of $\overline{D_1}$, $\overline{D_2}$ and (ii), we have $\partial(D_1, D_2) \cap S(\mu) = \emptyset$ and $w^*$ is the only fixed point of $\Phi(w, 0)$. It follows from Lemma 5.1 that

$$\deg(I - \Phi(w, 0), P, 0) = \text{index}(I - \mathcal{F}, w^*) = (-1)^0 = 1.$$

From (43) to (45), we have a contradiction. Thus, there exists a non-constant solution of model (8).

Since

$$\frac{1}{(a^2 + b^2)^2} + 1 + b > 2u_0^2 \Leftrightarrow \frac{d(c - d)}{2} > \sqrt{(b - 1)^2 - \frac{4c - d}{a^2}}$$

and when $D_2$ is large enough, then we have $(D_1 J_{22} + D_2 J_{21})^2 > 4D_1 D_2 (J_1 J_{22} - J_1 J_{21})$ and $0 < \mu^- (D_1, D_2) < \mu^*(D_1, D_2)$. Moreover,

$$\mu^- (D_1, D_2) \rightarrow 0, \mu^* (D_1, D_2) \rightarrow \frac{1}{D_1} (au_0^2 (1 + 2u_0^2) + \frac{b^2 u_0^2}{(a^2 + b^2)^2^2}),$$

as $D_2 \rightarrow \infty$.

Due to the fact that

$$\frac{1}{(a^2 + b^2)^2} \left(au_0^2 (1 + 2u_0^2) + \frac{b^2 u_0^2}{(a^2 + b^2)^2^2}\right) \in (\mu_j, \mu_{j+1})$$

for some $j \geq 1$, there exists $D_0 \gg 1$ such that

$$\mu^+ (D_1, D_2) \in (\mu_j, \mu_{j+1}), \quad 0 < \mu^+ (D_1, D_2) < \mu_j, \quad \forall D_2 \geq D_0.$$

Therefore, for large $D_2 > 0$, $\sum_{i=1}^{j} m(\mu_i)$ is odd which implies (i) and (ii) in the earlier arguments. This completes the proof. □
Acknowledgements

This research of F.R. is supported by Jiangsu Provincial Natural Science Foundation (BK 201404927) and Tianyuan Fund for Mathematics of NSF (11462132). The research of Y.K. is partially supported by NSF-DMS (1313312), Simons Collaboration Grants for Mathematicians (208902), NSF-IOS/DMS (1558127), The James S. McDonnell Foundation-UHC Scholar Award (220020472), and the research scholarship from College of Integrative Sciences and Arts, ASU.

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