THE INVERSE PROBLEM FOR COLLINEAR CENTRAL CONFIGURATIONS

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Abstract. We consider the problem: given a collinear configuration of n bodies, find the masses which make it central. We prove that for $n \leqslant 6$, each configuration determines a one-parameter family of masses (after normalization of the total mass). The parameter is the center of mass when n is even and the square of the angular velocity of the corresponding circular periodic orbit when n is odd. The result is expected to be true for any n.

Key words: inverse problem, n-body problem, central configuration

1. Introduction

Consider the Newtonian n-body problem:

$$\ddot{X}_i = A_i = \sum_{k \neq i} \frac{m_k(X_k - X_i)}{r_{ik}^3}, \quad i = 1, \ldots, n. \tag{1}$$

Here $m_i > 0$ are the masses of the bodies, $X_i \in \mathbb{R}^3$ are their positions, and $r_{ij} = |X_i - X_j|$. Let

$$C = m_1 X_1 + \cdots + m_n X_n, \qquad M = m_1 + \cdots + m_n, \qquad c = C/M \tag{2}$$

be the first moment, total mass and center of mass of the bodies.

The configuration $X_1, \ldots, X_n$ is called a *central configuration* if the acceleration vectors of the bodies satisfy:

$$A_i + \lambda(X_i - c) = 0, \quad i = 1, \ldots, n \tag{3}$$

for some constant λ. Such configurations give rise to simple, explicit solutions of the n-body problem [1]. If the bodies are placed in a central configuration and released with zero initial velocity, they will collapse homothetically to a collision at c. If the central configuration is planar, one can also choose initial velocities which lead to a periodic solution for which the configuration rigidly rotates around c with angular velocity $\sqrt{\lambda}$.

A collinear central configuration is called a *Moulton configuration* after F.R. Moulton who proved that for a fixed mass vector $m = (m_1, \dots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling) [2]. In this paper, we will be concerned with the inverse problem: given a collinear configuration, find the mass vectors, if any, for which it is a central configuration. In posing this question, we do not require $m_i > 0$.

Moulton also considered this question in [2]. His results depend on whether n is even or odd. In the even case $n = 2k$, Moulton introduced a certain Pfaffian, which we will call K_n. He found that if the center of mass is fixed at $c = 0$ and if $K_n \neq 0$ then there is a one parameter family of mass vectors for which the configuration is central. These mass vectors are all proportional to one another, so the mass ratios are actually unique.

In the odd case $n = 2k + 1$, Moulton found that when the center of mass was fixed at $c = 0$, the inverse problem could not always be solved. Only by imposing a constraint on the X_i could he obtain a corresponding mass vector. If the constraint is satisfied and if another Pfaffian is nonzero, he obtained a two-parameter family of masses so in this case the mass ratios are not unique.

The disparity between the even and odd cases is caused by fixing the center of mass. One goal of this paper is to show that when the center of mass is not fixed in advance, a duality between the even and odd cases emerges. In both cases, a given configuration determines a two-parameter family of masses making it central (provided certain Pfaffians are nonzero). The odd case is always solvable provided one chooses the correct center of mass, c. Having done this, we obtain a two-parameter family of masses as above. In the even case, the center of mass can be chosen arbitrarily. Each choice for c determines a one-parameter family of masses, as above, so there is again a two-parameter family in all.

Another interesting duality arises between the center of mass, $c = C/M$, and the ratio $\lambda' = \lambda/M$. In the odd case, c is uniquely determined and λ' can be taken as the parameter in the one-parameter family of mass ratios. In the even case, it is exactly the opposite, c is the parameter for the mass ratios and λ' is fixed. Our discussion in Sections 2 and 3 below will bring out the reasons for this.

All of these results are conditioned on the nonvanishing of various Pfaffians. These are all functions which depend only on the X_i. We conjecture that these functions are, in fact, nonzero for all noncollision configurations. In Section 4 we prove this for $n \leqslant 4$ even for quite general non-Newtonian potentials. We also describe computer assisted proofs for $n = 5, 6$ in the Newtonian case. Finally, for analytic potentials with sufficiently strong singularities, including the Newtonian case, we prove that the Pfaffians are nonzero for almost all configurations.

For purely mathematical reasons, it is natural when considering the inverse problem, to allow negative masses and potentials more general than the Newtonian one. One benefit of this is that our results apply fully to another well-known problem, namely the problem of relative equilibria of n collinear Helmholtz vortices (see [3]).

2. The Equation for Central Configurations

We will view a configuration of n bodies in $\mathbb{R}^d$ as a $n \times d$ matrix x with coefficients x_{ij}. The coordinates of the vector X_i in the introduction are then the x_{ij}, $1 \leqslant j \leqslant d$. The acceleration $a = \ddot{x}$ is a matrix of the same type. The vectors A_i of Equation (1) have coordinates a_{ij}, $1 \leqslant j \leqslant d$. Since each coefficient a_{ij} is a linear form in the masses, we can write $a_{ij} = \sum_k m_k \alpha_{kij}$. It is convenient to denote by α the *tensor* corresponding to the coefficients α_{kij}, and to consider also x, a and m as tensors, the coefficients of the last one being the m_i, $1 \leqslant i \leqslant n$. It will not be necessary here to distinguish between covariant and contravariant tensors, so all indices will be written as subscripts.

A key point for us is that α is antisymmetric in its two first indices. Indeed, it is clear from (1) that

$$\alpha_{kij} = r_{ik}^{-3}(x_{kj} - x_{ij}) = -\alpha_{ikj}, \quad \alpha_{iij} = 0. \tag{4}$$

As an exercise, one can deduce immediately from this antisymmetry that the acceleration $M^{-1} \sum_i m_i a_i$ of the center of mass c cancels.

Next we turn to the equation for central configurations. Their usual definition by (3) is not the correct one in the case $M = 0$. This case must be included in the discussion of the inverse problem however, since all choices for masses, including negative ones, are allowed. So we will generalize the definition a bit and call x a central configuration if and only if there exists a real number λ and a vector $\mu \in \mathbb{R}^d$ such that

$$A_i + \lambda X_i - \mu = 0, \quad i = 1, \ldots, n. \tag{5}$$

To make this equation a relation between tensors, we introduce a tensor L with all n coefficients $L_i = 1$. Then (5) becomes $a_{ij} + \lambda x_{ij} - L_i \mu_j = 0$. Replacing a_{ij} by its value above, this equation takes the form

$$\sum_k m_k \alpha_{kij} + \lambda x_{ij} - L_i \mu_j = 0. \tag{6}$$

In index free notation, the above equation becomes

$$m \lrcorner \alpha + \lambda x - L \otimes \mu = 0, \tag{7}$$

where $\lrcorner$ is a notation for a contracted or interior product, and $\otimes$ denotes the tensor product. As in the exercise above, let us contract the tensor m at the left of Equation (6) or (7). The first term drops out by antisymmetry and we obtain

$$\lambda C - M\mu = 0. \tag{8}$$

We can now check that (5) implies (3) if the total mass satisfies $M \neq 0$: we obtain $\mu = \lambda C/M = \lambda c$. The exotic case $M = 0$ has already been discussed in [5]. A new type of motion of relative equilibrium is possible with the extended definition of central configuration: when $M = \lambda = 0$, the particles travel together in a uniformly accelerated motion.

The rest of the paper is devoted to studying Equation (7) in the collinear case (see [6, 7] for some results in the planar case with $n = 4, 5$). Setting $d = 1$ produces several simplifications. The tensors x, a and α lose their last index. The quantity μ becomes a scalar and $L \otimes \mu$ becomes simply μL. Equation (7) is now

$$m \lrcorner \alpha + \lambda x - \mu L = 0. \tag{9}$$

The fact that α is antisymmetric will allow us to use exterior algebra as a computational device in the next section.

Rather than insisting on the Newtonian force law we will allow any α with $\alpha_{ij} = S_{ij}(x_i - x_j)$ where

$$S_{ij} = \varphi(r_{ij}^2)$$

and φ is a smooth real-valued function on some open subset of $\mathbb{R}$. The Newtonian case is given by $\varphi(s) = s^{-3/2}$, $s > 0$. In terms of the standard basis, $e_1, \ldots, e_n$ for $\mathbb{R}^n$ we have

$$\alpha = \sum_{i<j} \alpha_{ij}\, e_i \wedge e_j = \sum_{i<j} S_{ij}(x_i - x_j)\, e_i \wedge e_j.$$

A configuration x is called *nonsingular* if φ is well defined on it and if x is not proportional to L (this rules out a total collapse of all n bodies at the same position, even if φ is well-defined there). A nonsingular configuration will be called a *Moulton configuration with mass vector m* if there exist constants λ, μ for which (9) holds.

3. The Inverse Problem for Moulton Configurations

Let $x \in \mathbb{R}^n$ be a fixed, nonsingular configuration and let

$$S(x) = \{m \in \mathbb{R}^n : x \text{ is a Moulton configuration with mass vector } m\}.$$

Also define

$$S_{\lambda\mu}(x) = \{(m, \lambda, \mu) \in \mathbb{R}^{n+2} : (9) \text{ holds}\}$$
$$S_{\lambda}(x) = \{(m, \lambda) \in \mathbb{R}^{n+1} : (9) \text{ holds for some } \mu\}$$
$$S_{\mu}(x) = \{(m, \mu) \in \mathbb{R}^{n+1} : (9) \text{ holds for some } \lambda\}.$$

Since (9) is linear and homogeneous in (m, λ, μ), $S_{\lambda\mu}(x)$ is a subspace of $\mathbb{R}^{n+2}$. The natural projection maps of Euclidean spaces restrict to give a diagram of surjective maps:

$$\begin{tikzcd} & S_{\lambda\mu}(x) \arrow[dl] \arrow[d] \arrow[dr] & \\ S_\lambda(x) \arrow[dr] & {} \arrow[d] & S_\mu(x). \arrow[dl] \\ & S(x) & \end{tikzcd}$$

It follows that $S_\lambda(x)$ and $S_\mu(x)$ are subspaces of $\mathbb{R}^{n+1}$ and that $S(x)$ is a subspace of $\mathbb{R}^n$. Moreover

PROPOSITION 1. *If x is nonsingular then the spaces $S(x)$, $S_{\lambda\mu}(x)$, $S_\lambda(x)$, $S_\mu(x)$ are all isomorphic and have dimension at least 2.*

Proof. It suffices to show that the projection $\pi\colon S_{\lambda\mu}(x) \to S(x)$ is injective. Suppose $v = (m, \lambda, \mu)$ is in the kernel of π. Then $m = 0$ and so (9) gives $\lambda x - \mu L = 0$. Since x is not a multiple of L we must have $\lambda = 0$. Then we also have $\mu = 0$ and so $v = 0$ as required. It is clear that $\dim S_{\lambda\mu}(x) \geqslant 2$ and so the same is true for the other spaces. □

In particular, this means that every nonsingular configuration, x, is a central configuration for some choice of masses (this is false in the noncollinear case, $d > 1$; see [6, 7]). Moreover, the masses which make x a central configuration are not unique. The freedom of rescaling m is obvious, but the proposition shows that there is still at least one free parameter in the set of mass ratios making x central. Our goal is to find conditions which guarantee that $\dim S(x) = 2$ and to identify natural parameters for the plane $S(x)$.

First we study the question of dimension. Equation (9) shows that $S_{\lambda\mu} = \ker {}^tB$ where B is the $(n+2) \times n$ matrix:

$$B = \begin{bmatrix} \alpha \\ \hline x \\ \hline -L \end{bmatrix}$$

and α is the $n \times n$ antisymmetric matrix representing the two-form of the same name. The following result uses exterior algebra to determine the rank of B.

PROPOSITION 2. *Let $\alpha^k = \alpha \wedge \cdots \wedge \alpha$ denote the k-fold wedge product of the two-form α. For any configuration $x \in \mathbb{R}^n$, and for any natural number k, we have*

$$\mathrm{rk} B \leqslant 2k - 1 \iff (\alpha^k = 0 \quad \textit{and} \quad \alpha^{k-1} \wedge x \wedge L = 0),$$
$$\mathrm{rk} B \leqslant 2k \iff (\alpha^k \wedge L = 0 \quad \textit{and} \quad \alpha^k \wedge x = 0).$$

Proof. The rank is the dimension of the vector space generated by the $n+2$ row vectors. Consider the vector space generated by the n first rows, and choose a base $v_1, \dots, v_{2m}$ of this vector space. Its dimension, $2m$, is the rank of α, and is characterized by the conditions $\alpha^m \neq 0$, $\alpha^{m+1} = 0$. We also know that $\alpha^m = bv_1 \wedge \cdots \wedge v_{2m}$ for some real number $b \neq 0$. The integer m being known, rk$B$ may take only three values: either $2m$ if $\alpha^m \wedge L = \alpha^m \wedge x = 0$, or $2m+1$ when the previous condition is not satisfied and when $\alpha^m \wedge L \wedge x = 0$, or finally $2m+2$ if the last condition is not satisfied. The assertion of the proposition follows easily, considering the different values of m compatible with each of the four hypotheses. □

To express the conditions for $\dim S(x) = 2$, let us introduce the two 'Pfaffian functions' $K_n(x)$ and $K_n^{xL}(x)$ in the even case $n = 2k$, and the two Pfaffian functions $K_n^x(x)$ and $K_n^L(x)$ in the odd case $n = 2k+1$. In the standard base $e_1, \dots, e_n$ of $\mathbb{R}^n$, the formulas are

$$\begin{aligned} &\alpha^k = K_n(x)\, e_1 \wedge \cdots \wedge e_n, \qquad &&\alpha^{k-1} \wedge x \wedge L = K_n^{xL}(x)\, e_1 \wedge \cdots \wedge e_n, \\ &\alpha^k \wedge L = K_n^L(x)\, e_1 \wedge \cdots \wedge e_n, \qquad &&\alpha^k \wedge x = K_n^x(x)\, e_1 \wedge \cdots \wedge e_n. \end{aligned} \tag{10}$$

Propositions 1 and 2 then give immediately:

THEOREM 1. *Let x be a nonsingular configuration of n bodies. If n is even, then* $\dim S(x) = 2$ *if and only if $K_n(x)$ or $K_n^{xL}(x)$ is nonzero. If n is odd then* $\dim S(x) = 2$ *if and only if $K_n^x(x)$ or $K_n^L(x)$ is nonzero.*

We conjecture that for the Newtonian potential, $K_n(x)$, $K_n^{xL}(x)$ and $K_n^L(x)$ are actually nonzero for all nonsingular configurations x, and hence, we always have $\dim S(x) = 2$. The next section is devoted to some partial results in this direction. The fourth function $K_n^x(x)$ is excluded because it is generally not translation invariant. Translation corresponds to replacing x by $x + \tau L$ and from the definitions we have $K_n^x(x + \tau L) = K_n^x(x) + \tau K_n^L(x)$. If $K_n^L(x)$ is in fact nonzero, then there will be some translate with $K_n^x(x + \tau L) = 0$.

We turn now to the question of choosing parameters for $S(x)$. We will see that each of our four criteria of Theorem 1 leads to a parametrization using two of the four quantities λ, μ, C, M.

The simplest result comes from writing (9) as a nonhomogeneous linear equation for $m \in \mathbb{R}^n$:

$$m \lrcorner \alpha = \mu L - \lambda x. \tag{11}$$

If $K_n(x) \neq 0$ then α is a nondegenerate two-form and this equation can be uniquely solved for m as a linear function of (μ, λ). So Equation (11) leads to a parametrization of the space $S(x)$ in terms of two of our four parameters.

To handle the other three cases we will derive new equations similar in form to (11) which will lead to parametrizations of the spaces $S_{\lambda\mu}(x)$, $S_\lambda(x)$ and $S_\mu(x)$.

By Proposition 1, these project to parametrizations of $S(x)$. The new equations are obtained by combining (9) with the equations of Equation (2) defining the first moment, C, and the total mass, M.

Extend the standard basis, $e_1, \ldots, e_n$ of $\mathbb{R}^n$ to a basis for $\mathbb{R}^{n+2}$ by adding two new vectors e_λ and e_μ. Define a two-form on $\mathbb{R}^{n+2}$ by

$$\alpha_{\lambda\mu} = \alpha - x \wedge e_\lambda + L \wedge e_\mu. \tag{12}$$

Then the equation

$$(m, \lambda, \mu) \lrcorner \alpha_{\lambda\mu} = -Ce_\lambda + Me_\mu \tag{13}$$

is equivalent to (9) together with the definitions of C and M. If $\alpha_{\lambda\mu}$ is nondegenerate, we obtain a parametrization of $S_{\lambda\mu}(x)$ with parameters C and M.

Now $\alpha_{\lambda\mu}$ is nondegenerate if and only if $\alpha_{\lambda\mu}^{k+1} \neq 0$. Expanding this power using (12), we have

$$\alpha_{\lambda\mu}^{k+1} = \frac{(k+1)k}{2}\, \alpha^{k-1} \wedge x \wedge L \wedge e_\lambda \wedge e_\mu = \frac{(k+1)k}{2}\, K_n^{xL}(x)\; e_1 \wedge \cdots \wedge e_\mu.$$

So if $K_n^{xL}(x) \neq 0$ we have a parametrization in terms of C and M.

To handle the odd cases, define two-forms on $\mathbb{R}^{n+1}$

$$\alpha_\lambda = \alpha - x \wedge e_\lambda, \qquad \alpha_\mu = \alpha + L \wedge e_\mu.$$

Then the equation

$$(m, \lambda) \lrcorner \alpha_\lambda = \mu L - Ce_\lambda \tag{14}$$

is equivalent to (9) together with the definition of C. Nondegeneracy of α_λ leads to a parametrization of $S_\lambda(x)$ by μ and C. On the other hand, the equation

$$(m, \mu) \lrcorner \alpha_\mu = Me_\mu - \lambda x \tag{15}$$

is equivalent to (9) together with the definition of M. Nondegeneracy of α_μ leads to a parametrization of $S_\mu(x)$ by M and λ.

If $n = 2k+1$ then $n+1 = 2(k+1)$, and so these forms will be nondegenerate provided

$$\alpha_\lambda^{k+1} = -(k+1)\,\alpha^k \wedge x \wedge e_\lambda \neq 0, \qquad \alpha_\mu^{k+1} = (k+1)\,\alpha^k \wedge L \wedge e_\mu \neq 0.$$

These criteria involve the quantities $K_n^x(x)$, $K_n^L(x)$, respectively.

Thus if one of the four Pfaffians is nonzero $S(x)$ can be linearly parametrized by one of the pairs (λ, μ), (C, M), (μ, C) or (λ, M).

Next we will see that relation (8) implies that, in contrast to the pairs of variables used as parameters, other pairs must maintain a constant ratio on $S(x)$. First note that because of Equation (2) and Proposition 1, the quantities λ, μ, C, M are linear forms on $S(x)$. Writing (8) as $\lambda C = \mu M$ we have two factorizations of a quadratic polynomial into linear factors. Unique factorization implies that the factors which appear on each side must be the same up to constants.

If (λ, μ) are the parameters, they are independent linear forms on $S(x)$, and we must have $C = r\mu$ and $M = r\lambda$ for some constant r. On the other hand, if (C, M) are the parameters, we have $\mu = rC$ and $\lambda = rM$. Similar results are deduced if either (μ, C) or (λ, M) can be used as parameters. As the case $r = 0$ is not excluded so far, it is convenient to express the previous relations in term of projective classes. We will denote the projective class of a nonzero element $(\lambda, \mu) \in \mathbb{R}^2$ by $[\lambda, \mu] \in \mathbb{RP}^1$. We can now summarize our results.

THEOREM 2. *Let x be a nonsingular, collinear configuration of n bodies. If n is even then $S(x)$ can be linearly parametrized by (λ, μ) if $K_n(x) \neq 0$ or by (C, M) if $K_n^{xL}(x) \neq 0$. In both cases, $[\lambda, M]$ and $[\mu, C]$ are constant on $S(x)$. If n is odd then $S(x)$ can be linearly parametrized by (μ, C) if $K_n^x(x) \neq 0$ or by (λ, M) if $K_n^L(x) \neq 0$. In both cases, $[C, M]$ and $[\lambda, \mu]$ are constant on $S(x)$.*

The quantities $[C, M]$ and $[\lambda, M]$ are perhaps more interesting than the others since they represent the center of mass c and the ratio $\lambda' = \lambda/M$, respectively. These can be used as parameters of the projective version of $S(x)$, i.e. the projective line $[S(x)]$, when $K_n^{xL}(x) \neq 0$ or $K_n^L(x) \neq 0$. Since these inequalities hold at least generically, the main results of this section can be expressed as follows. A fixed collinear configuration of n bodies, x, will generally be central for a one-parameter family of mass values $(m_1, \ldots, m_n)$ after normalizing $M = m_1 + \cdots + m_n = 1$. If n is even, each choice of such masses gives the configuration a different center of mass but all of the resulting circular periodic orbits will have the same angular velocity. On the other hand, when n is odd, the different choices of masses determine the same center of mass, but different angular velocities.

One may wonder whether the preferred parameters $[C, M]$ and $[\lambda, M]$ could be used even when the required Pfaffians vanish. The following curious result can be used to show that this is not the case.

PROPOSITION 3. *Let $x \in \mathbb{R}^n$ be any nonsingular configuration, and $m \in \mathbb{R}^n$ a set of masses in $S(x)$. If $n = 2k$ is even, then the matrix*

$$\begin{bmatrix} \lambda & \mu & K_n(x) \\ M & C & kK_n^{xL}(x) \end{bmatrix}$$

is of rank at most one. If n is odd, then the matrix

$$\begin{bmatrix} \lambda & \mu \\ M & C \\ K_n^L(x) & K_n^x(x) \end{bmatrix}$$

is of rank at most one.

Proof. We have to check the nullity of three 2×2 determinants in the even case, and again three in the odd case. As we already know (8), four determinants remain. We first note that by (11)

$$m \lrcorner \alpha^k = k(m \lrcorner \alpha) \wedge \alpha^{k-1} = \mu k L \wedge \alpha^{k-1} - \lambda k x \wedge \alpha^{k-1}.$$

If $n = 2k$, then the $(n+1)$-form $L \wedge \alpha^k = 0$ and so

$$0 = m \lrcorner (L \wedge \alpha^k) = \langle m, L \rangle \alpha^k - L \wedge (m \lrcorner \alpha^k) = M\alpha^k + \lambda k\, L \wedge x \wedge \alpha^{k-1}.$$

By (10) this implies $MK_n - \lambda k K_n^{xL} = 0$ and this is the nullity of one of the determinants. Now in the same way,

$$0 = m \lrcorner (x \wedge \alpha^k) = C\alpha^k - \mu\, k\, x \wedge L \wedge \alpha^{k-1},$$

which gives $CK_n - \mu k K_n^{xL} = 0$. This completes the even case.

In the odd case, we compute

$$0 = m \lrcorner (L \wedge x \wedge \alpha^k) = Mx \wedge \alpha^k - CL \wedge \alpha^k$$

and

$$0 = m \lrcorner \alpha^{k+1} = (k+1)(\mu L - \lambda x) \wedge \alpha^k.$$

Expanding the second equation and using (10) completes the proof. □

Now suppose that for some configuration we have $K_n(x) \neq 0$ but $K_n^{xL}(x) = 0$. Then the entire second row of the first matrix in Proposition 3 must vanish. For this configuration we will have $C = 0$ and $M = 0$ for all choices of the masses $m \in S(x)$. Thus, $[C, M]$ is not defined and cannot be used as a parameter in $[S(x)]$. Note, however, that if there is at least one set of positive masses in $S(x)$, this case cannot occur.

Similarly, if $K_n^x(x) \neq 0$ but $K_n^L(x) = 0$, the first column of the second matrix must vanish and so we have $\lambda = M = 0$. We cannot use $[\lambda, M]$ as a parameter in this case.

We close this section with some examples. The main conclusions of Theorem 2 – that c is a free parameter and λ' is fixed when n is even, and that λ' is free and c fixed when n is odd – are nicely illustrated by the simplest cases $n = 2, 3$ for the Newtonian potential.

For $n = 2$ every configuration is equivalent by translation and scaling to the configuration with $x_1 = 0$ and $x_2 = 1$. We have

$$x = e_2, \qquad L = e_1 + e_2, \qquad \alpha = -e_1 \wedge e_2,$$
$$\alpha^{k-1} \wedge x \wedge L = x \wedge L = -e_1 \wedge e_2.$$

Hence $K_2(x) = K_2^{xL}(x) = -1 \neq 0$. Equation (11) becomes

$$m_2 e_1 - m_1 e_2 = \mu e_1 + (\mu - \lambda) e_2,$$

and so, we obtain a parametric solution for the masses making x a central configuration

$$m_1 = \lambda - \mu, \qquad m_2 = \mu.$$

Of course this means that any set of masses will work. The first moment and total mass are by (2):

$$C = m_2 = \mu, \quad M = m_1 + m_2 = \lambda,$$

and so, we have a parametrization in terms of the (projective) center of mass

$$m_1 = M - C, \quad m_2 = C.$$

Also note that $\lambda = M$ so $\lambda' = \lambda/M = 1$ is constant.

Next, consider three bodies with positions $x_1 = 0$, $x_2 = 1$, and $x_3 = 1+r$ where $r > 0$. Up to translation and scaling, this is the general three-body configuration. We have

$$x = e_2 + (1+r)e_3, \qquad \alpha = -e_1 \wedge e_2 - \frac{1}{(1+r)^2}\, e_1 \wedge e_3 - \frac{1}{r^2}\, e_2 \wedge e_3,$$

$$\alpha^k \wedge x = \frac{1-(1+r)^3}{(1+r)^2}\, e_1 \wedge e_2 \wedge e_3,$$

$$\alpha^k \wedge L = -\frac{1+2r+r^2+2r^3+r^4}{r^2(1+r)^2}\, e_1 \wedge e_2 \wedge e_3.$$

Hence

$$K_3^x(x) = \frac{1-(1+r)^3}{(1+r)^2}, \qquad K_3^L(x) = -\frac{1+2r+r^2+2r^3+r^4}{r^2(1+r)^2},$$

and both are nonzero for all $r > 0$. We have

$$\alpha_\mu = \alpha + (e_1 + e_2 + e_3) \wedge e_\mu,$$

and Equation (15) becomes

$$\begin{aligned}&[m_2 + (1+r)^{-2} m_3 - \mu]\, e_1 + (-m_1 + r^{-2} m_3 - \mu)\, e_2 + \\ &\quad + [-(1+r)^{-2} m_1 - r^{-2} m_2 - \mu]\, e_3 + \\ &\quad + (m_1 + m_2 + m_3)\, e_\mu = M e_\mu - \lambda e_2 - \lambda(1+r) e_3.\end{aligned}$$

This leads to a parametric solution for the masses in terms of λ and M:

$$m = \frac{\big((1+r)^2(M-\lambda r^3),\, r^2(\lambda(1+r)^3 - M),\, r^2(1+r)^2(M-\lambda)\big)}{1+2r+r^2+2r^3+r^4}.$$

Remarkably, all of these masses yield the same center of mass

$$c = \frac{r^3(3+3r+r^2)}{1+2r+r^2+2r^3+r^4}.$$

We also note that it is always possible to choose the parameters so that all three masses are positive. We need only take

$$(1+r)^{-3} \leqslant \frac{\lambda}{M} \leqslant \max(1, r^{-3}).$$

In the next section we will see that this is still true for much more general potentials provided $n = 3$. On the other hand one can show that for $n \geqslant 4$, it is generally not possible to arrange that all masses be positive, even in the Newtonian case. We checked, for example, that the configuration $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $x_4 = 4$ does not admit positive masses.

4. Study of the Pfaffians

In this section we will prove that the functions $K_n(x)$, $K_n^{xL}(x)$, and $K_n^L(x)$, $n \leqslant 4$, are nonzero for all nonsingular collinear configurations provided the function φ defining the force law satisfies certain conditions. In addition we will describe a computer aided study of the cases $n = 5, 6$ for the Newtonian case where $\varphi(s) = s^{-3/2}$.

For $n = 1$ we have $x = x_1 e_1$ and $L = e_1$. Since x is a multiple of L, every configuration is singular and our claim is vacuous. Nevertheless, we have $K_1^L(x) = 1$. On the other hand, $K_1^x(x) = x_1$, illustrating the lack of translation invariance.

For $n = 2$, we have

$$\alpha = S_{12}(x_1 - x_2)\, e_1 \wedge e_2, \qquad x \wedge L = (x_1 - x_2)\, e_1 \wedge e_2$$

and so

$$K_2(x) = \varphi(r_{12}^2)(x_1 - x_2) \quad \text{and} \quad K_2^{xL}(x) = x_1 - x_2.$$

Thus $K_2^{xL}(x)$ is nonzero for all nonsingular configurations. The same is true for $K_2(x)$ provided that $\varphi(s) \neq 0$ for $s \neq 0$.

For $n = 3$, computation of $\alpha \wedge L$ shows that

$$K_3^L(x) = (x_2 - x_3)S_{23} + (x_3 - x_1)S_{13} + (x_1 - x_2)S_{12}.$$

PROPOSITION 4. *If the function φ is strictly decreasing, then $K_3^L(x)$ is strictly negative when $x_1 < x_2 < x_3$.*

Proof. We can rearrange the formula to get

$$K_3^L = (x_2 - x_3)(S_{23} - S_{13}) + (x_1 - x_2)(S_{12} - S_{13}).$$

The assumed order implies that S_{13} is the smallest of the S_{ij}. □

Exactly the same hypothesis on φ guarantees the existence of a positive mass vector. To see this, note that for $n = 3$, the possible masses form a two-dimensional plane in $\mathbb{R}^3$. An equation for this plane can be found by taking the wedge product of (9) with $\gamma = x \wedge L$ to obtain $(m \lrcorner \alpha) \wedge \gamma = 0$. This equation takes the form

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ \gamma_{23} & \gamma_{31} & \gamma_{12} \\ \alpha_{23} & \alpha_{31} & \alpha_{12} \end{vmatrix} = 0.$$

This equation for Euler configurations appears almost in this form in [8], §358. There are always positive masses in the hyperplane if and only if the above linear form in the masses does not take three values of the same sign on the three vectors e_1, e_2 and e_3. But those three values are: $\gamma_{31}\gamma_{12}\big(\varphi(s_{12}) - \varphi(s_{31})\big)$, $\gamma_{12}\gamma_{23}\big(\varphi(s_{23}) - \varphi(s_{12})\big)$ and $\gamma_{23}\gamma_{31}\big(\varphi(s_{31}) - \varphi(s_{23})\big)$. Now $\gamma_{ij} = x_i - x_j$, so making the convention $x_1 < x_2 < x_3$, and supposing φ strictly decreasing, the first value is negative, and the third is positive. So *there always exists positive masses.*

For $n = 4$ we have

$$\begin{aligned} K_4(x) &= 2(x_1 - x_2)(x_3 - x_4)S_{12}S_{34} - 2(x_1 - x_3)(x_2 - x_4)S_{13}S_{24} + \\ &\quad + 2(x_1 - x_4)(x_2 - x_3)S_{14}S_{23}, \\ K_4^{xL}(x) &= (x_1 - x_2)(x_3 - x_4)(S_{12} + S_{34}) - \\ &\quad -(x_1 - x_3)(x_2 - x_4)(S_{13} + S_{24}) + (x_1 - x_4)(x_2 - x_3)(S_{14} + S_{23}). \end{aligned}$$

First we show that $K_4(x) \neq 0$ for power law forces.

PROPOSITION 5. *Let* $\varphi(s) = s^w$ *with* $w \in \mathbb{R}$. *If* $x_1 < x_2 < x_3 < x_4$, *then* $K_4(x) < 0$ *if* $w > 0$, $K_4(x) > 0$ *if* $w < 0$.

Proof. Put $A = (x_1 - x_2)(x_3 - x_4)$, $B = (x_1 - x_3)(x_2 - x_4)$ and $C = (x_1 - x_4)(x_2 - x_3)$ to obtain

$$\tfrac{1}{2}K_4 = A^{2w+1} - B^{2w+1} + C^{2w+1}.$$

Note that $B = A + C$, so a classical inequality applies. □

Next we consider $K_4^{xL}(x)$ making use of the following alternative formula:

$$K_4^{xL}(x) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ s_{12} + s_{34} & s_{13} + s_{24} & s_{14} + s_{23} \\ S_{12} + S_{34} & S_{13} + S_{24} & S_{14} + S_{23} \end{vmatrix} \tag{16}$$

where $s_{ij} = r_{ij}^2 = (x_i - x_j)^2$.

PROPOSITION 6. *If* φ *is a strictly convex function, then* $K_4^{xL}(x)$ *is strictly positive when* $x_1 < x_2 < x_3 < x_4$.

Proof. The assumed order on the x_i implies a partial ordering of the s_{ij}. In particular, we have

$$s_{12} < s_{13} < s_{14}, \quad s_{23} < s_{13} < s_{14}, \quad s_{23} < s_{24} < s_{14}, \quad s_{34} < s_{24} < s_{14}.$$

We now make

$$A_0 = \begin{vmatrix} 1 & 1 & 1 \\ s_{12} & s_{13} & s_{14} \\ S_{12} & S_{13} & S_{14} \end{vmatrix}, \quad A_1 = \begin{vmatrix} 1 & 1 & 1 \\ s_{23} & s_{13} & s_{14} \\ S_{23} & S_{13} & S_{14} \end{vmatrix},$$

$$A_2 = \begin{vmatrix} 1 & 1 & 1 \\ s_{23} & s_{24} & s_{14} \\ S_{23} & S_{24} & S_{14} \end{vmatrix}, \quad A_3 = \begin{vmatrix} 1 & 1 & 1 \\ s_{34} & s_{24} & s_{14} \\ S_{34} & S_{24} & S_{14} \end{vmatrix}.$$

The convexity of φ implies that these four numbers are strictly positive. They and K_4^{xL} are also unchanged if we change the $S_{ij} = \varphi(s_{ij})$ to $\bar{S}_{ij} = \bar{\varphi}(s_{ij})$, where $\bar{\varphi}$ is a new function differing from φ by an affine function: $\bar{\varphi}(s) = \varphi(s) + as + b$, where a and b are arbitrary real numbers. We choose these two constants such that $\bar{S}_{14} = \bar{S}_{23} = 0$, and we first find $A_1 = (s_{23} - s_{14})\bar{S}_{13}$ and $A_2 = (s_{23} - s_{14})\bar{S}_{24}$. This shows that $\bar{S}_{13}$ and $\bar{S}_{24}$ are negative numbers. We then find

$$\bar{S}_{12}(s_{14} - s_{13}) = A_0 + \bar{S}_{13}(s_{14} - s_{12}), \quad \bar{S}_{34}(s_{14} - s_{24}) = A_3 + \bar{S}_{24}(s_{14} - s_{34}).$$

This gives

$$\begin{aligned} K_4^{xL} &= p_0(A_0 + \bar{S}_{13}(s_{14} - s_{12})) + p_3(A_3 + \bar{S}_{24}(s_{14} - s_{34})) - \\ &\quad - (x_1 - x_3)(x_2 - x_4)(\bar{S}_{13} + \bar{S}_{24}), \end{aligned}$$

where

$$p_0 = \frac{(x_1 - x_2)(x_3 - x_4)}{s_{14} - s_{13}} = \frac{x_1 - x_2}{2x_1 - x_3 - x_4}, \qquad p_3 = \frac{x_3 - x_4}{x_1 + x_2 - 2x_4}.$$

The coefficient of $\bar{S}_{13}$ in this expression is

$$\begin{aligned} q_1 &= p_0(x_2 - x_4)(2x_1 - x_2 - x_4) - (x_1 - x_3)(x_2 - x_4) \\ &= (2x_1 - x_3 - x_4)^{-1}(x_2 - x_4)(x_2 - x_3)(x_2 + x_3 + x_4 - 3x_1). \end{aligned}$$

This is a negative number. The coefficient of $\bar{S}_{24}$ is

$$q_2 = (x_1 + x_2 - 2x_4)^{-1}(x_1 - x_3)(x_2 - x_3)(3x_4 - x_1 - x_2 - x_3),$$

and is also negative. Now $K_4^{xL} = p_0A_0 + p_3A_3 + q_1\bar{S}_{13} + q_2\bar{S}_{24}$ is the sum of four positive numbers. □

Unfortunately, this sequence of results about general potentials does not continue so simply. It is not even clear what hypotheses should be made on the function φ to assure that $K_5^L(x) \neq 0$. We turn now to some computer assisted results for the Newtonian case.

For small values of n it is possible to use a computer algebra system to carry out the wedge products defining the Pfaffians, at least for simple power law potentials.

If this is done for the Newtonian potential one obtains rational functions of the mutual distances. Assuming the ordering $x_1 < \cdots < x_n$ and expressing all mutual distances in terms of the quantities $x_{i+1} - x_i$ one obtains a rational function in $n-1$ variables, all positive.

For example, when $n = 4$ we set $x = x_2 - x_1$, $x_3 - x_2 = 1$ and $z = x_4 - x_3$ and obtain $K_4^{xL}(x)$ as a rational function with numerator

$$\begin{aligned} &x^7z^4 + 3x^6z^5 + 3x^5z^6 + x^4z^7 + 2x^7z^3 + 11x^6z^4 + 18x^5z^5 + 11x^4z^6 + \\ &+ 2x^3z^7 + x^7z^2 + 13x^6z^3 + 37x^5z^4 + 37x^4z^5 + 13x^3z^6 + x^2z^7 + \\ &+ 2x^7z + 8x^6z^2 + 32x^5z^3 + 54x^4z^4 + 32x^3z^5 + 8x^2z^6 + 2xz^7 + \\ &+ x^7 + 10x^6z + 19x^5z^2 + 36x^4z^3 + 36x^3z^4 + 19x^2z^5 + 10xz^6 + z^7 + \\ &+ 4x^6 + 18x^5z + 18x^4z^2 + 18x^3z^3 + 18x^2z^4 + 18xz^5 + 4z^6 + \\ &+ 6x^5 + 14x^4z + 6x^3z^2 + 6x^2z^3 + 14xz^4 + 6z^5 + 4x^4 + 4x^3z + \\ &+ 4xz^3 + 4z^4 + x^3 + z^3 \end{aligned}$$

and denominator

$$x^2\,(x+1)^2\,(x+1+z)^2\,z^2\,(1+z)^2\,.$$

Remarkably, every term in these polynomials carries a plus sign. This gives an alternative proof that $K_4^{xL}(x) > 0$ for all x in the Newtonian case.

Using this technique, we find $K_4(x) > 0$, $K_5^L(x) > 0$, $K_6(x) < 0$ and $K_6^{xL}(x) < 0$ for all nonsingular configurations. For example, the numerator of $-K_6^{xL}(x)$ is a polynomial in five positive variables with 15158 coefficients, all positive! (After writing it to a file, a text editor was used to search for minus signs.)

Although we cannot prove the analogous result for arbitrary n we can prove something for generic configurations. If $\varphi(s)$ is a real analytic function for $s > 0$ then each of our four functions is real analytic on the set of nonsingular configurations in $\mathbb{R}^n$. The complement of the zero set of such a function is open and dense and has full Lebesgue measure provided that the function does not vanish identically. The following result shows that this is true for certain types of potential, including the Newtonian case.

PROPOSITION 7. *If $r|\varphi(r^2)| \to \infty$ as $r \to 0_+$ then for every n the functions $K_n(x)$, $K_n^{xL}(x)$, $K_n^x(x)$ and $K_n^L(x)$ are not identically zero.*

Proof. We will consider the case of $K_n^L(x)$ for $n = 2k+1$, the other cases being similar. The proof is by induction on k. We have already discussed $K_1^L(x)$ and $K_3^L(x)$ and found them to be nonzero for all x. Let $k \geqslant 1$ and assume the result is known for $n = 2k-1$.

Consider the family of configurations $x(r)$ where $x_1 < \cdots < x_{2k}$ are fixed and $x_{2k+1} = x_{2k} + r$, $r > 0$. By the inductive hypothesis we may choose the x_i so that

$$K_{2k-1}^L(x_1, \ldots, x_{2k-1}) \neq 0.$$

By definition, $K^L_{2k+1}(x(r))$ is the coefficient of $e_1 \wedge \cdots \wedge e_{2k+1}$ in the wedge product $\alpha^k \wedge L$. All of the terms of α have bounded coefficients as $r \to 0_+$ except the term

$$S_{(2k)(2k+1)}(x_{2k} - x_{2k+1})e_{2k} \wedge e_{2k+1} = -r\varphi(r^2)e_{2k} \wedge e_{2k+1}.$$

Considering how this term will appear in the wedge product $\alpha^k \wedge L$ we find that

$$K^L_{2k+1}(x(r)) = -kC(x)r\varphi(r^2) + \mathrm{O}(1)$$

where $C(x)$ is the coefficient of $e_1 \wedge \cdots \wedge e_{2k-1}$ in the wedge product $\alpha^{k-1} \wedge L$. Since the terms of α which involve x_{2k} and x_{2k+1} do not contribute to $C(x)$ we have

$$C(x) = K^L_{2k-1}(x_1, \ldots, x_{2k-1}) \neq 0.$$

It follows that $|K^L_{2k+1}(x(r))| \to \infty$ as $r \to 0_+$ and, in particular, $K^L_{2k+1}(x)$ is not identically zero. □

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Note Added in Proof

A. Chenciner pointed out to us a paper of H. E. Buchanan, 'On certain determinants connected with a problem in Celestial Mechanics', *Bull. Amer. Math. Soc.* **5** (1909), 227–231, which claims to prove our conjecture that the function $K_n(x)$ is nonzero for all configurations, x. The argument is incorrect. The zeros of the nth derivatives considered on pages 228 and 229 may tend to infinity with n.

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