Central Configurations—A Problem for the Twenty-first Century

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What does it take to qualify as a "problem for the twenty-first century?" Obviously, the topic must be of fundamental importance, it must be difficult to solve as manifested by the number of excellent people who have tried—but failed, and non-experts should be able to appreciate the basic issues. The particular problem described here, which involves finding certain geometric configurations that arise in several places including the Newtonian N-body problem, easily satisfies these criteria. The issue is not new: a couple of centuries ago Euler and Lagrange answered the question for the three-body problem. But the full problem for $N \geq 4$ has proved resistant to solution. What makes the topic so important is that even partial answers add to our understanding of N-body systems. These brief comments already suggest why Smale [17] listed one of the problems described below as a mathematical challenge for the twenty-first century. Of particular interest, this topic is one where non-experts can be expected to add new insights. In fact, beyond describing aspects of this fascinating issue, a goal of this paper to enlist new people to this area.

The general objective is to find all of the *central configurations* and their basic properties. While the term may be new for you, you probably have seen some of them if your interests include paddling a canoe, or watching the weather channel, as these are the configurations that tend to be formed by the swirly vortices—or cyclones in the Indian Ocean— as they move along. My examples from *celestial mechanics* (the mathematical study of how N-astronomical bodies move when governed by Newton's equations) will indicate why these configurations are so important.

To motivate the definition, start with the equations of motion for the Newtonian N-body problem

$$m_j\mathbf{r}_j'' = \frac{\partial U}{\partial \mathbf{r}_j} = \sum_{k\neq j} \frac{m_k m_j(\mathbf{r}_k - \mathbf{r}_j)}{r_{j,k}^3}, j = 1, \dots, N, \qquad U = \sum_{j<k} \frac{m_j m_k}{r_{j,k}}, \tag{1}$$

where m_j and $\mathbf{r}_j$, are, respectively, the mass and position vector (relative to the center of mass) for the j^{th} particle, and $r_{jk} = |\mathbf{r}_j - \mathbf{r}_k|$ is the distance between particles j and k. Rather than analyzing this notoriously difficult equation, it would be much easier if we could justify dealing with the special setting where

$$\mathbf{r}_j'' = \lambda(t)\mathbf{r}_j, \quad j = 1, \dots, N, \tag{2}$$

where λ is a scalar function that is the same for all particles. The first step is to determine what it means if *at even one instant* of time Eq. 2 holds. By combining Eqs. 1, 2, it follows that Eq. 2 holds iff the configuration formed by the bodies satisfies

$$\lambda\mathbf{r}_j = \frac{1}{m_j}\frac{\partial U}{\partial \mathbf{r}_j}, \quad j = 1, \dots, N. \tag{3}$$

A configuration that satisfies Eq. 3 at an instant of time is called a *central configuration.* In other words, a central configuration occurs when the acceleration vector for each particle lines up with its force vector, and the scalar difference is the same for all particles. In this special setting, the dynamics of each particle mimics that of a "central force" problem.

1 Where do they occur?

What makes central configurations important is that they arise so often in N-body systems. To indicate why this is so, start with colliding particles. To simplify the story, assume that all particles collide at the center of mass at time $t = 0$.

A reasonable guess about the motion of the colliding particles (as they approach the origin) is that they behave like $\mathbf{r}_j \sim \mathbf{A}_j t^\alpha$ where we need to determine the vector constant $\mathbf{A}_j$ and scalar α. *If* we are allowed to twice differentiate this asymptotic relationship,[1] we would have $\mathbf{r}_j'' \sim (\alpha-1)\alpha\mathbf{A}_j t^{\alpha-2}$. Substituting terms into the equations of motion (Eq. 1) leads to

$$(\alpha-1)\alpha\mathbf{A}_j t^{\alpha-2} \sim \frac{1}{m_j}\sum_{k\neq j}\frac{m_k m_j(\mathbf{A}_k-\mathbf{A}_j)}{|\mathbf{A}_j-\mathbf{A}_k|^3}t^{-2\alpha}, \quad j=1,\ldots,N.$$

By comparing terms (e.g., this requires $t^{\alpha-2} = t^{-2\alpha}$), we have that if these steps can be justified, then $\alpha = \frac{2}{3}$ and the $\mathbf{A}_j$ terms satisfy the central configuration Eq. 3; that is, colliding particles *form a central configuration in the limit as $t \to 0$!* Proving that all of this actually happens requires justifying that collision orbits can be described in terms of $\mathbf{r}_j \sim \mathbf{A}_j t^\alpha$ and that the asymptotic relationship can be differentiated. Siegal [15] did this for triple collisions in the three-body problem, Wintner [18] did this for the complete collapse of all N-bodies (i.e., all bodies collide at the center of mass), and I did this for all possible kinds of collisions [12, 14]; e.g., it may be that only some particles collide where, say, a five-body collision occurs at one place while, simultaneously, a ten-body collision occurs elsewhere.

Now jump from collisions to expansions. Here our interest is in the evolution of the N-body problem: can we find the general behavior of all possible N-body systems as $t \to \infty$? Such a description exists (Saari [10], Marchal and Saari [5]), and it shows that particles tend to separate from one another in three different ways. By using this information, a rough description of the evolution of an universe emerges. Start by calling those collections of particles that tend to remain relatively close to one another "galaxies." Then the galaxies form "groups of galaxies" that separate from one another at a specified rate, and finally the groups of galaxies separate from one another like a multiple of time. It is interesting how this mathematical result corresponds to what we actually observe; e.g., our home galaxy, the Milky Way, belongs to the "Local Group."

Part of the analysis requires determining how the galaxies within a particular "group of galaxies" separate from one another. To develop insight we might guess that the j^{th} galaxy separates from the center of mass of the group like $\mathbf{R}_j \sim \mathbf{A}_j t^\alpha$. By mimicking what was done for collisions, where the asymptotic relationship is differentiated and then substituted back into the equations of motion (but here the differentiation is even more problematic), we discover that the galaxies should separate like $\mathbf{R}_j \sim \mathbf{A}_j t^{2/3}$ and tend to form *central configurations.* In other words, expect the galaxies within a group to eventually create well defined, expanding configurational shapes. This is what I proved; again, I had to establish that there are settings where $\mathbf{R}_j \sim \mathbf{A}_j t^\alpha$ holds and that it is permitted to differentiate the asymptotic relations.

Between the extremes of where particles collide or expand is where they move in the plane behaving like a rigid body. By use of complex variables, the position of each particle can be expressed as $z_j(t) = a_j e^{\omega i t}$ where ω is a constant. There are no mathematical worries in this setting about whether it is permissible to differentiate, so we have that $z_j''(t) = -\omega^2 a_j e^{\omega i t}$. Substituting the appropriate terms back into the equations of motion, we learn that if the particles rotate like

[1] While we always can integrate an asymptotic relations, in general we cannot differentiate it. For instance, although $f(t) = t^2(1 + t\sin(\frac{1}{t^4})) \sim t^2$ as $t \to 0$, it is *not* true that $f'(t) \sim 2t$. Tauberian theorems, as developed by Hardy and Littlewood [3] (also see Saari [14]), describe when an asymptotic relations can be differentiated.

a rigid body, then, again, they must form a *central configuration* where $\lambda = -\omega^2$.

I could go on to describe what happens when surfaces defined by the constants of the system (the total energy and the angular momentum) bifurcate either in the topological (Smale [16]) or geometric frameworks (Saari [13]), but you probably already suspect that these bifurcation locations tend to be characterized by central configurations. Indeed, in any limiting or extreme setting[2] where the motion is delicately determined by the equations of motion, expect a central configuration to occur. In other words, as discovered through research in this area, central configurations play a particularly "central" role in the study of N-body systems.

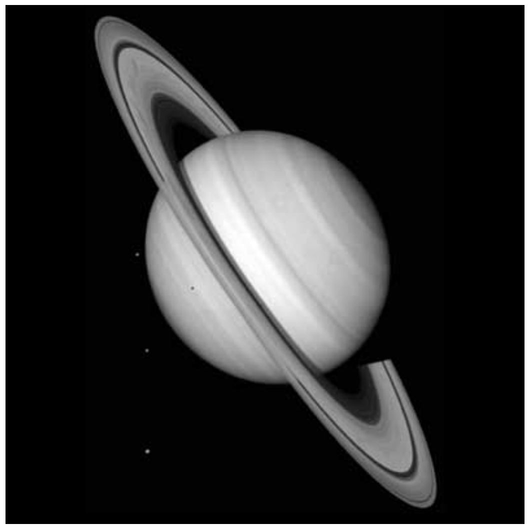


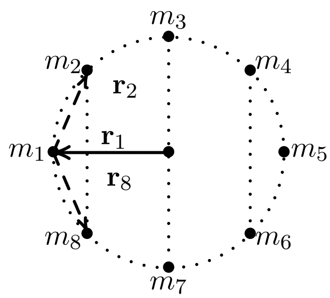
Fig. 1. Saturn and its rings; my thanks to NASA/JPL-Caltech for permission to use this picture.

2 Other applications?

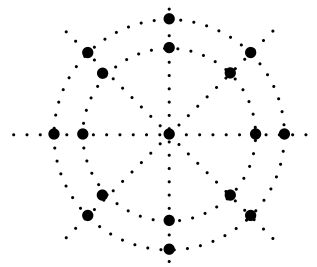
The above description suggests other settings where we now might expect to find central configurations. For instance, the rings of Saturn (Fig. 1) appear to involve "rigid body" rotating motion with particles on a circle, so it seems reasonable to try to analyze them via rigid body motions. In arguing that the rings had to be individual particles, Maxwell introduced this approach back in the 1850s. The connection with the above discussion is that a rigid body motion requires the particles to form a central configuration. A first step, then, is to determine whether it is possible to create a central configuration by symmetrically placing particles on a circle.

It is possible, and the idea is illustrated with Fig. 2a where equal masses are symmetrically positioned on a circle. The advantage of using equal masses in a symmetric configuration is that the force acting on any one body determines what happens to the force with all other bodies. Thus we just need to analyze what happens with one of them, say m_1. Partition the remaining bodies into pairs that are the same distance from m_1; e.g., m_2 and m_8 define one pair. It could be that one body, as true with m_5 in Fig. 2a, cannot be placed in a pair. But this creates no problem because with m_5 directly opposite from m_1, its gravitational force on m_1 is directed toward the center of mass.

[2] This is in the collinear or coplanar problems; different configurations arise in higher dimensional physical spaces. For instance, see Saari [13] or Palmore [8].



a. Central configuration



b. More rings

Fig. 2. Central configurations for Saturn

To determine the force on m_1 defined by the $\{m_2, m_8\}$ pair, divide the force that m_2 exerts on m_1—the dashed arrow in Fig. 2a indicating that m_1 is being pulled directly toward m_2— into components as indicated in Fig. 2a. Namely, find the horizontal component of this force (in the $-\mathbf{r}_1$ direction) and the vertical component (on the dotted line passing through m_2). A similar decomposition applies to the force coming from m_8 except that the force in the vertical direction is precisely the opposite of the m_2 vertical force. As such, the two forces components that are along the dotted line cancel: this means that each pair combines to create a force on m_1 that is directed toward the center of mass $\mathbf{0}$. By symmetry, the same argument shows that the force on each particle is directed toward $\mathbf{0}$ with equal magnitude. All that remains is to compute the common scalar value of λ so that $\lambda \mathbf{r}_j = \mathbf{r}_j''$. Consequently, the particles form a central configuration. Of course, because each particle is pulled toward the center of mass, λ has a negative value.

We now have the ring, but where is Saturn? No problem: a similar symmetry argument shows that by placing a new particle m_{N+1}, at the center of mass, the forces over all particles cancel so that $\mathbf{r}_{N+1}'' = \mathbf{r}_{N+1} = \mathbf{0}$. As $\lambda \mathbf{0} = \mathbf{0}$, this new setting also is a central configuration. Because this assertion holds for *all choices of the mass* m_{N+1}, Saturn can have its actual mass, and the ring particles can have very small masses. We need the masses on the ring to have small values to analyze whether this rigid body motion is stable, but that is a different unsolved research problem.

There is an obvious objection: while Maxwell's approach defines a ring, Fig. 1 clearly displays that Saturn has several rings. We could dismiss this problem by arguing that the masses in a ring are so small that they do not affect other rings. In addition to being dubious, this argument runs counter to our goal of understanding central configurations. So let me introduce a trick (described in Saari [14]) that allows us to construct cental configurations with as many rings as desired.

The idea is illustrated in Fig. 2b where particles on the inner and outer rings are symmetrically positioned on spokes coming from the center: the particles on the inner ring, with radius 1 from the origin, all have mass m while those on the outer ring, which has radius $r > 1$, all have mass m^*. The above symmetry arguments prove that the force acting on each particle is along the spoke passing through the particle and the origin. But the magnitude, and even the direction, of the forces on each ring can differ. An easy way to see this is to consider what happens should the outer ring be nearly on top of the inner one. The inverse square force law requires a particle's neighbor on the spoke to create the dominant component of force; as the force on each body is directed toward its partner, one force vector must point away from the center of mass. By placing the particles close enough together, (so $r - 1$ has a very small value) the force can be made arbitrarily large.

Symmetry ensures that what happens to one particle on a ring happens to all of them. Thus there is a λ_1 so that all particles on the inner ring satisfy $\lambda_1 \mathbf{r}_j = \mathbf{r}_j''$. The particles on the outer ring satisfy a similar relationship with a λ_2. Moreover, both λ_j values depend continuously on the r value. For instance, by letting $r \to 1$, we have that $\lambda_1(r) \to \infty$. (The force on each inner ring body is infinitely large and points toward its neighboring partner on a spoke, or outwards.) But, as $r \to \infty$, which means that the second ring is pushed so far away that its particles have minimal effect on the inner ring, $\lambda_1(r)$ approaches the negative value of the Fig. 2a one-ring setting.

The other function, $\lambda_2(r)$, approaches $-\infty$ as $r \to 1$. (The force is infinitely large and points toward the neighboring particle on the inner ring, or toward the origin.) But, as $r \to \infty$, we have that $\lambda_2(r) \to 0$. (With particles being far apart, the force and $\mathbf{r}_j''$ terms have small values. On the other hand, the $\mathbf{r}_j$ distance is very large, so λ_2 in $\lambda_2 \mathbf{r}_j = \mathbf{r}_j''$ must have a small value to attain equality.) Thus $\lambda_1(r)$ and $\lambda_2(r)$ define two continuous curves that, from their properties, must cross. At such a crossing point $\lambda_1(r) = \lambda_2(r)$, which defines a central configuration. Obvious modifications of this argument hold for any number of rings. Here is a challenge: what modifications to this argument lead to other classes of central configurations? (Some suggestions are in Saari [14], while others are immediate. For instance, instead of a circle, use a sphere.)

3 Finding λ, and deciding how to count

To find still other central configurations, let

$$R^2 = \sum_{j=1}^{N} m_j \mathbf{r}_j^2. \tag{4}$$

Notice that R is a crude measure of the radius of the system of N-bodies. Now consider the RU product of the radius R and self-potential U of the system; I call RU the *configurational measure*. The connection this measure has with central configurations can be seen by using the product rule from differential calculus to obtain

$$\nabla RU = R\left(\frac{U}{R^2} m_1 \mathbf{r}_1 - \frac{\partial U}{\partial \mathbf{r}_1}, \dots, \frac{U}{R^2} m_N \mathbf{r}_N - \frac{\partial U}{\partial \mathbf{r}_N}\right). \tag{5}$$

We can anticipate the following theorem by comparing the components of this gradient with Eq. 3.

Theorem 1 *A central configuration is defined if and only if $\nabla RU = \mathbf{0}$. Moreover, at a central configuration, $\lambda = -\frac{U}{R^2}$.*

I don't know who first proved this theorem, but simple proofs can be found in the classic book Wintner [18], or in Chapter 2 of my book [14]. Notice how this expression, which is basic for what is discussed in the rest of this paper, separates the study of central configurations from actual N-body dynamics. This separation, which emphasizes the critical point structure of an analytic function, is why I stated earlier that non-experts in celestial mechanics can become experts in the study of central configurations.

The configurational measure also indicates how to count the number of central configurations. But first, recall that the center of mass is fixed at the origin; e.g., $\sum m_j \mathbf{r}_j = \mathbf{0}$. Thus, letting $M = \sum m_j$ be the total mass of the system, we have that

$$R^2 = \frac{1}{M} \sum_{j<k} m_j m_k r_{j,k}^2. \tag{6}$$

Establishing Eq. 6 just involves carrying out the multiplication

$$\sum_{j<k} m_j m_k (\mathbf{r}_j - \mathbf{r}_k)^2 = \frac{1}{2} \sum_j \sum_k m_j m_k [\mathbf{r}_j^2 + \mathbf{r}_k^2 - 2\mathbf{r}_j \cdot \mathbf{r}_k] = \frac{1}{2}[2MR^2 + \sum_j m_j \mathbf{r}_j \cdot \sum_k m_k \mathbf{r}_k].$$

The conclusion follows because each $\sum_j m_j \mathbf{r}_j$ term in the last expression equals $\mathbf{0}$.

A consequence of this exercise in dot products is that the configurational measure depends only on the mutual distances between particles. In other words, it is *rotation invariant* because any change in the orientation of the configuration does not affect the RU value. Even more, R is homogeneous of degree 1; i.e., multiplying all distances by the positive scalar μ leads to a μR value. Because U is homogeneous of degree -1, we have that RU is homogeneous of degree 0: changes in the size of the configuration do not effect the RU value. To see where I am headed, recall from high school geometry that Euler similarity classes equate those configurations that can be obtained from each other with changes in scale and rotation. This means that the value of the configurational measure RU is fixed over an Euler similarity class. In particular, if $\nabla RU = \mathbf{0}$ for a particular configuration, than after any rotation or scalar change in the configuration, we still have $\nabla RU = \mathbf{0}$. Consequently, any rotation or scalar change of a central configuration remains a central configuration. So *when counting central configurations, we count the Euler equivalence classes.*

We now come to one of Smale's problems for the 21^{st} century, which renews interest in a conjecture that probably was formulated in the nineteenth century and is described in Wintner [18].

> *It is believed that for any value of N and any choices of the positive masses m_j, there are only a finite number of central configurations. Is this true?*

To describe this problem in terms of critical point theory, recall that, generically, an analytic function has its critical points separated. This generic setting corresponds to the belief that, once the invariances captured by the Euler similarity classes are divided out, RU always will have separated critical points and, hence, a finite number of central configurations. The question, then, is equivalent to showing that RU does not have a line of critical points outside an Euler similarity class. Can this happen? In the eighteenth century, Euler and Lagrange proved for $N = 3$ that there are only four central configurations. (I describe them below.) With a computer assisted proof, Hampton and Moeckel [2] proved that the four-body problem admits only a finite number. Roberts [9], on the other hand, proved that such a continuum *does* exist if we allow negative masses. Not much is known beyond this.

4 Weighted means

We could appeal to the well known strategy that if a mathematical problem proves to be difficult, then a solution may become apparent by considering a more general problem. To motivate the extended setting that I recommend for central configurations [14], recall the standard calculus problem where the student is asked to find the rectangle, with unit area, that has the smallest perimeter. It is easy to show that the answer is a square, but can this be done without calculus?

To do so, start with the inequality

$$xy = (\frac{x+y}{2})^2 - (\frac{x-y}{2})^2 \leq (\frac{x+y}{2})^2, \tag{7}$$

where equality holds iff $x = y$. Because xy is the area of a rectangle with leg lengths x and y, while $2(x+y)$ is the perimeter, it follows from the extreme ends of this expression that the minimum length of the perimeter is $4(xy)^{\frac{1}{2}} = 2(x+y)$ where the minimum occurs iff the leg lengths are equal iff the rectangle is a square.

4.1 Weighted means

The extreme ends of Eq. 7 corresponds to a special case of a general class of *weighted mean* inequalities that have been studied, in various forms, for centuries. To introduce the terms, let $\mathbf{a} = (a_1, \ldots, a_J)$ be a J-vector of positive terms—these are the variables. The "weights" are fixed positive values $g_1, \ldots, g_J$.

Definition 1 *For $p \neq 0$, the p^{th} weighted mean of $\mathbf{a}$ is*

$$\mathcal{W}_p(\mathbf{a}) = (\frac{\sum g_i a_i^p}{\sum g_i})^{1/p}. \tag{8}$$

The geometric mean, which is the definition of the weighted mean for $p = 0$, is

$$\mathcal{W}_0(\mathbf{a}) = (a_1^{g_1} a_2^{g_2} \ldots a_J^{g_J})^{1/\sum g_i}.$$

The following result, which has been known for over a century (see the delightful classic by Hardy, Littlewood, and Polya [4]), will identify other central configurations:

Theorem 2 *For $p_1 > p_2$, we have that*

$$\mathcal{W}_{p_1}(\mathbf{a}) \geq \mathcal{W}_{p_2}(\mathbf{a}) \tag{9}$$

where equality holds if and only if $\mathbf{a} = a(1, , 1, \ldots, 1)$.

Equation 7, then, is the special case of Thm. 2 where $g_1 = g_2 = 1$, $J = 2$, $p_1 = 1$ and $p_2 = 0$. (This relationship relates the *arithmetic* ($p_1 = 1$) and *geometric* ($p_2 = 0$) means.) An extension of Thm. 2 shows that $a_1 = a_2 = \ldots = a_J$ is the only critical point of $\mathcal{W}_{p_1}(\mathbf{a})/\mathcal{W}_{p_2}(\mathbf{a})$.

To relate all of this to central configurations, choose the g weights to be $m_j m_k$ and the a values to be $r_{j,k}$: these choices make R a multiple of the quadratic mean $\mathcal{W}_2$ and U^{-1} a multiple of the harmonic mean $\mathcal{W}_{-1}$. In other words, the study of central configurations becomes a special case of the study of weighted means. Viewing the problem in this manner makes it easier to find certain central configurations. As an example of what is gained, Thm. 2 means that the only critical point of RU, which is a scalar multiple of $\mathcal{W}_2/\mathcal{W}_{-1}$, is when $r_{1,2} = r_{1,3} = \ldots = r_{N-1,N}$, or when the particles form an equilateral object. For three particles, the configuration is an equilateral triangle, for four it is an equilateral tetrahedron, for more than four particles, the equilateral figure cannot exist in our $\mathbb{R}^3$ physical space, but it can in $\mathbb{R}^d$, $d > 3$. While the equilateral triangle and tetrahedron configurations have been known for centuries, the standard proofs are laborious: by using weighted means, they are immediate.

A surprising fact is that the equilateral triangle, which is the only non-collinear three-body central configuration, is a central configuration no matter what the values of the masses—even in extreme settings where one mass could be a rock while the other two could be, say, a planet and the Sun. This actually occurs.

4.2 Trojans

Way back in 1772 and by using very different techniques, Lagrange discovered that these vertices of an equilateral triangle create a central configuration. It took another 134 years until the German astronomer Max Wolf checked whether this configuration actually occurs in our solar system: it does. If you are going to search for such a configuration, the natural choice is to use the two heaviest bodies, the Sun and Jupiter, to form the defining leg of an equilateral triangle. This defines two equilateral triangles—one leads and the other follows Jupiter. The surprise is that Wolf discovered asteroids located at the remaining vertex of each triangle! Wolf called them the *Trojans.* Jumping ahead to 1990, the asteroid *Eureka* was found at the triangular point of Sun-Mars, and there may be objects in the triangular locations of the Sun-Earth and other planets (Saari [14]). Indeed, as noted by several, the triangular points defined by the Earth and our moon would make a nice docking position for a satellite. Central configurations matter!

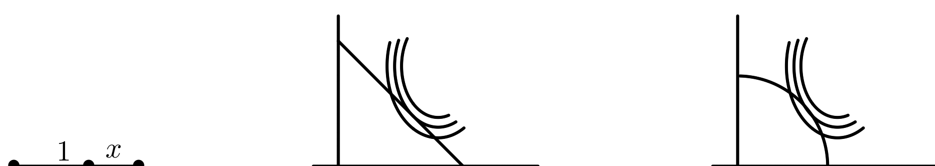
4.3 Collinear configurations

The next central configurations are the collinear ones—this is where all particles lie on a straight line. Euler discovered the collinear configurations for the three-body problem and Moulton [7] analyzed the general N-body case where he proved the following:

Theorem 3 *(Moulton [7]) The N body problem with positive masses, there are precisely $N!/2$ collinear central configurations. More precisely for each way the particles can be ordered along a line, there is a unique position that causes a central configuration.*

According to this result, there is unique position between the Earth and the Moon where a object could be placed—and (theoretically) it would remain there forever.[3] The other two locations have the Earth in the middle or the Moon in the middle. Indeed, for the three-body problem, we now have all of the central configurations—the three collinear central configurations and the equilateral triangle. To indicate how Euler found the collinear configurations, as in Fig. 3, set one distance equal to unity and the other with length x. Substituting these distances into the equations for a central configuration leads to a fifth order polynomial in x; the sole positive zero of this polynomial is the x value needed for a central configuration.

If it takes a fifth order polynomial in x to handle the three-body problem, you can imagine the complexity of the equations in at least two variables that would arise for the four-body problem, and then the even increased complexity for the five, and six, and . . . body problems. In other words, to prove his result, Moulton needed to develop a new strategy: he found what happened each time a new mass was introduced. (See [7, 14].) But even this analysis is complex. So, perhaps as a way to entice non-experts to join this search for central configurations, let me show how a particularly elementary proof (Saari [11, 14]) emerges by using the weighted means.



a. Collinear **b.** Linear constraint **c.** Concave down constraint

Fig. 3 Collinear central configurations

First consider the problem of describing all critical points of a function f restricted to a linear constraint. To make the problem easier, suppose that the level sets of f are strictly concave up. (See Fig. 3b.) The answer is obvious for any dimensional space; the unique critical point occurs where the level set of f kisses the linear constraint. Neighboring level sets either miss the linear constraint or meet it in more than one point. An even easier problem is when the linear constraint is replaced by a constraint surface that is strictly concave down (Fig. 3c). The answer, which obviously is the same, shows how to find collinear central configurations.

[3]Of more practical value, this position has proved to be a useful location for "mid-course" corrections.

As shown earlier, the central configurations that result strictly from the $\mathcal{W}_2/\mathcal{W}_{-1}$ structure are the equilateral central configurations. Thus all remaining central configurations involve finding the critical points of $\mathcal{W}_2/\mathcal{W}_{-1}$ subject to natural constraints that are imposed on the distances between particles. These constraints determine whether or not the distances can be used to create a configuration. To illustrate with the collinear three-body problem, where all particles are on a straight line and m_2 is in the middle, if we have $r_{1,2} = r_{2,3} = 1$, then we must have $r_{1,3} = 2$. Namely, to allow the r_{jk} distances to define a configuration that lies on a line, the degenerate triangle inequality of $r_{1,2} + r_{2,3} - r_{1,3} = 0$ must be satisfied.

So, finding N-body collinear central configurations is equivalent to finding the critical point of $\mathcal{W}_2/\mathcal{W}_{-1}$ subject to all of the $r_{jk} + r_{kl} - r_{jl} = 0$ type linear constraints that are needed to specify that all of the particles are on a line; the actual choices of constraints represent a specified ordering of the bodies on this line. Because $\mathcal{W}_2/\mathcal{W}_{-1}$ is homogeneous of degree zero, we can include $\mathcal{W}_2$ with the linear constraints by setting it equal to unity: the equivalent problem now is to describe the critical points of $1/\mathcal{W}_{-1}$ subject to these constraints. The proof follows from the nature of the constraints: they are linear or concave down (the $\mathcal{W}_2 = 1$ constraint). In contrast, the level sets of $1/\mathcal{W}_{-1}$, or of $U = \sum \frac{m_j m_k}{r_{jk}}$ in a space of mutual distances, are concave up. The above geometric argument depicted by Fig. 3c now shows that there is a unique central configuration for each ordering of the particles on the line. This proves Moulton's theorem.

5 Other central configurations

Other than special cases, such as where some masses have very small values, or all masses are equal, or ..., not many other central configurations are known. So in this section, a sense of what else is possible is given. The description is built around the notion that a way to find central configurations with four or more particles in a plane, or five or more in the plane or in a general three dimensional space, we need to find the critical points of $\mathcal{W}_2/\mathcal{W}_{-1}$ subject to the constraints that the distances between particles must satisfy in order to allow a configuration to be constructed.

As it turns out, handling these constraints constitutes the full and real source of the complexity of the central configuration problem. Thus, insight about new ways to handle these constraints probably will lead to new conclusions. Even more: this description shows that the problems about central configurations are special cases of the following more general mathematical issue:

Characterize all critical points when $\mathcal{W}_{p_1}/\mathcal{W}_{p_2}$ is restricted to algebraic surfaces.

For the remainder of this paper, I show how simple geometry can lead to new conclusions about central configurations. As an illustration, it is reasonable to expect that choices of masses can be found so that the three four-body configurations of Fig. 4 are central configurations. If so, for what masses? Before reading more, let me ask the reader to speculate what might be the answer.

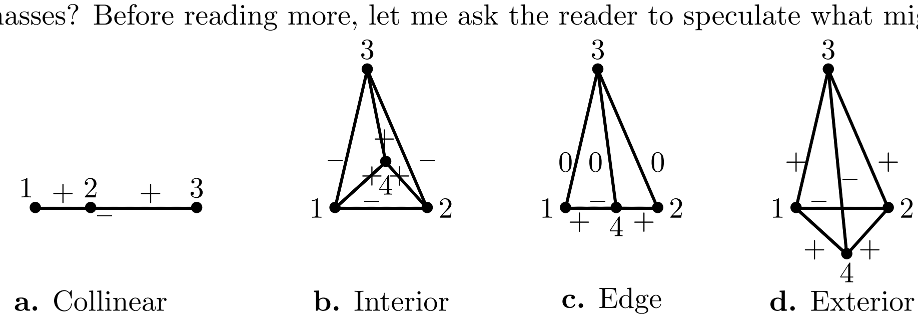


Fig. 4. Signs for gradients

Let $\mathbb{R}^{\binom{N}{2}}$ represent mutual distances between particles. To avoid confusion between actual r_{jk} distances between particles, let the variables be $\{\xi_{i,j}\}_{i<j}$; thus $\xi_{i,j}$ represents $r_{i,j}$ only when the physical constraints, given by $g_k(\xi) \geq 0$, are satisfied. Defining

$$\tilde{R} = [\frac{1}{M}\sum_{i<j} m_i m_j \xi_{i,j}^2]^{1/2}, \quad \tilde{U} = \frac{m_i m_j}{\xi_{i,j}},$$

and letting $g_k(\xi) = 0$ represent the physical constraints needed for the ξ variables to create a constructible object of the correct dimension, a central configuration is found by the expression

$$\nabla \tilde{R}\tilde{U} = \sum_k \lambda_k \nabla g_k. \tag{10}$$

The standard approach for coplanar four-bodies, which traces its origin to the work of Dziobeck [1], is to define the constraint equation g as the formula for the volume of a tetrahedron: thus $g = 0$ corresponds to a degenerate tetrahedron, which is a coplanar four-body configuration. While insightful, the computations required by this approach have proved to be difficult and messy, so efforts have been made to simplify the analysis; e.g., see Moeckel [6].

But the volume formula is not the only approach; e.g., as introduced in [14], other choices are to use areas or angles. To illustrate with the degenerate tetrahedron of Fig. 4b, another choice is

$$\angle 314 + \angle 412 - \angle 312,$$

which is negative for an object that cannot be constructed, positive for a tetrahedron, and zero for a coplanar figure (a degenerate tetrahedron). Another choice would be the sum of the areas of the three small triangles minus the area of the larger triangle: it is zero for a coplanar, degenerate tetrahedron.

Again, by using only elementary geometry, it is possible to find new results while avoiding the complications of selecting a specific choice for g. (This material comes from Chap. 3 of [14], so details can be found there.) For example, if only one constraint $g = 0$ is needed, then a way to find partial results is to use what I call "The Rule of Signs." The idea is to use the Lagrange multiplier Eq. 10 by comparing the signs of the components of ∇g and $\nabla \tilde{R}\tilde{U}$ where we avoid specifically choosing and computing ∇g.

Let me illustrate the idea with the constraint equation $\xi_{12} + \xi_{23} - \xi_{13} = 0$ for the collinear setting of Fig. 4a (which is Fig. 3.6 from [14]). Rather than computing the partial derivative with respect to ξ_{12}, notice that if the ξ_{12} length is increased, while the other two leg lengths are kept fixed, the Fig. 4a construction must buckle to create a triangle. From this and the definition of the partial derivative, we have that $\frac{\partial g}{\partial \xi_{12}} > 0$. On the other hand, increasing the ξ_{13} length while holding the other two fixed would tear the object apart: to create a triangle (where $g > 0$), we need to decrease ξ_{13}, so $\frac{\partial g}{\partial \xi_{13}} < 0$. The signs of the three partials of ∇g are indicated in Fig. 4a.

The same argument applies to the degenerate tetrahedrons where some particle is in the convex hull of the other three as depicted in Fig. 4b. To create an actual tetrahedron (where $g > 0$) by changing leg lengths, we must increase the lengths of any of the three interior legs, or decrease the lengths or any of the three exterior legs. Thus the signs of the $\frac{\partial g}{\partial \xi_{jk}}$ partials are as indicated in Fig. 4b. The reader can carry out a similar analysis to verify the signs of components for ∇g for a Fig. 4d configuration. The Fig. 4c choices follow from continuity and comparing Figs. 4b, d.

We now know the signs of the ∇g components for all four-body coplanar configurations, so it remains to find the signs of the $\nabla \tilde{R}\tilde{U}$ components. A straightforward computation shows that

$$\frac{\partial \tilde{R}\tilde{U}}{\partial \xi_{jk}} = A m_j m_k [\frac{\tilde{U}}{M\tilde{R}^2}\xi_{jk} - \frac{1}{\xi_{jk}^2}] = A m_j m_k \xi_{jk} [\frac{1}{\xi_{CAL}^3} - \frac{1}{\xi_{jk}^3}] \tag{11}$$

where A is a common term and I call $\xi_{CAL} = [M\tilde{R}^2/\tilde{U}]^{1/3}$ the *configurational average length.* It follows immediately from Eq. 11 that the ξ_{jk} component of $\nabla \tilde{R}\tilde{U}$

- is negative iff $\xi_{jk} < \xi_{CAL}$,
- equals zero iff $\xi_{jk} = \xi_{CAL}$, and
- is positive iff $\xi_{jk} > \xi_{CAL}$,

Results about the kinds of configurations that can, and cannot, be central configurations follow immediately just by comparing Eq. 11 with the signs of ∇g given in Fig. 4. As $\lambda < 0$ in Eq. 10, a positive sign for a ξ_{jk} leg in Fig. 4 means that for a central configuration, $\xi_{jk} < \xi_{CAL}$, etc.

Theorem 4 *(Saari [14]) A four-body coplanar central configuration, where one particle is in the convex hull defined by the other three, must have the maximum length of the three inner legs strictly smaller than the minimum length of the three outer legs. On the other hand, if no particle is in the hull of the other three, then the maximum length of the four outer legs must be strictly smaller than the shorter of the two interior diagonals. There never can be a non-collinear four-body central configuration where three of the particles are on straight line.*

This theorem answers the above question about whether Figs. 4 b, c, d could be central configurations for some choices of masses. We now know the answer: none of them can be central configurations. This is because in Fig. 4b, the exterior leg r_{12} is shorter than the interior leg r_{34}, in Fig. 4c three particles are on a straight line, and in Fig. 4d, diagonal r_{12} is shorter than exterior leg r_{13}. Figure 4c cannot be a central configuration because each of the three legs with a zero (for ∇g) by it would have to have the same leg length of ξ_{CAL}, and that is physically impossible.

A message coming from Thm. 4 is that, whatever the mass values, these central configurations must be surprisingly regular in shape. The same kind of result holds for five-body non-coplanar central configurations in $\mathbb{R}^3$ where the g constraint now is in terms of a degenerate pentahedron. I will leave it as a challenge to the reader to assign signs for the different ξ_{jk} legs; with thought, it is not overly difficult.

Much more is possible; e.g., are there simple ways to obtain information about the five, or six body coplanar central configurations? Also, notice that the terms of ∇g depend only on the leg lengths; they are independent of the masses. Can this be fully exploited to give us more information about central configurations? While there are many other options and directions using these approaches. if results obtained in this manner are not described in [14], they probably are new.

6 Summary

The issue of understanding central configurations has been with us for centuries, yet so little is known. While answers for these questions will be of value for celestial mechanics, it is reasonable to expect that people from outside of this academic area will make important advances. For instance, if a continuum of central configurations does exist, then there exist values of masses so that a algebraic portion of the level set of $\tilde{R}\tilde{U}$ agrees with the constraint sets: this appears to be a problem from real algebraic geometry. I also expect that a useful way to consider these issues is to examine the more general question of weighted means and the critical points of $\mathcal{W}_{p_1}/\mathcal{W}_{p_2}$ when constrained to surfaces. My sense is that understanding the more general problem will uncover mathematical structures that will allow gains in analyzing central configurations.

Moreover, as I tried to illustrate, it is possible for simple geometric insights to lead to new results. I encourage you to join us in analyzing these fascinating questions!

References

[1] Dziobek, O., Uber einen merkwurdigen fall des vielkorperproblems, *Astr. Nachr.* **152** (1900), 33-46.

[2] Hampton, M., and R. Moeckel, Finiteness of relative equilibria of the four-body problem, University of Minnesota preprint, July, 2004.

[3] Hardy, G. H. and Littlewood, J. E. *Quart. J. Math.* **46** (1915), 215-219.

[4] G. Hardy, Littlewood, J. E., and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1934.

[5] Marchal, C. and D. G. Saari, On the final evolution of the n-body problem, *J. Diff. Equations* **20** (1976), 150–186.

[6] Moeckel, R., Generic finiteness for Dziobek configurations, *Transactions of AMS*, **353** (2001), 4673-4686.

[7] Moulton, F. R., The straight line solutions of the problem of N-bodies, *Annal of Math.* Second Series, **12** (1910), 1-17.

[8] Palmore, J., Central configurations and relative equilibria of the N-body problem in E^4, *Celestial Mechanics,* **21** (1980), 21-24.

[9] Roberts, G., A continuum of relative equilibria in the five-body problem, *Physica D* **127** (1999), 141-145.

[10] Saari, D. G., Expanding gravitational systems, *Trans. Amer. Math. Soc.* **156** (1971), 219–240.

[11] Saari, D. G., On the role and the properties of central configurations in the n-body problem, *Celestial Mechanics* **21** (1980), 9–20.

[12] Saari, D. G., Manifold structure for collisions and for hyperbolic-parabolic orbits in the n-body problem, *J. Diff. Eqs.* **55** (1984), 300–329.

[13] Saari, D. G., From rotations and inclinations to zero configurational velocity surfaces II: The best possible configurational velocity surfaces, *Celestial Mechanics* **40** (1987), 197–223.

[14] Saari, D. G., *Collisions, Rings, and Other Newtonian N-Body Problems*, American Mathematical Society, Providence, R.I., 2005.

[15] Siegel, C. L., Der Dreierstoss, *Ann. Math.* **42** (1941), 127-168.

[16] Smale, S., Topology and mechanics, II. The planar n-body problem. *Inventiones Math.* **11**, (1970) 45-64.

[17] Smale, S., Mathematical problems for the next century, in *Mathematics: Frontiers and Perspectives*, ed. V. Arnold, M. Atiyah, P. Lax, and B. Mazur, American Math. Soc. 2000, 271-294.

[18] Wintner, A., *The Analytic Foundations of Celestial Mechanics*, Princeton University Press, 1941.